On the Number of Plane Graphs

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Abstract

We investigate the number of plane geometric, i.e., straight-line, graphs, a set S of n points in the plane admits. We show that the number of plane graphs and connected plane graphs as well as the number of cycle-free plane graphs is minimized when S is in convex position. Moreover, these results hold for all these graphs with an arbitrary but fixed number of edges. Consequently, we provide simple proofs that the number of spanning trees, cycle-free graphs (forests), perfect matchings, and spanning paths is also minimized for point sets in convex position.

In addition we construct a new extremal configuration, the so-called double zig-zag chain. Most noteworthy this example bears $\Theta^*(\sqrt{72}^n) = \Omega^*(8.4853^n)$ triangulations and $\Theta^*(41.1889^n)$ plane graphs (omitting polynomial factors in both cases), improving the previously known best maximizing examples.

1 Introduction

Let us denote by $\mathcal{S}_n$ the set of sets of n points in the plane in general position, that is, no three points of a set $S \in \mathcal{S}_n$ lie on a common line. With $\Gamma_n \in \mathcal{S}_n$ we denote any set of n points in convex position. Throughout this paper we consider *plane geometric graphs* G on top of $S \in \mathcal{S}_n$. That means that the set of vertices of G is S, edges of G are straight-line segments spanned by vertices of S and two edges of G do not intersect in their interior but might have endpoints in common. From now on we use the term graph to denote plane geometric graphs, unless otherwise noted.

In other words, we consider the rectilinear drawing of the complete graph K_n with vertex set $S \in \mathcal{S}_n$ and study its crossing-free subgraphs. The problem of how large the number of such subgraphs may be has been attracting a lot of attention; many references can be found in the handbook [15] and in the lately published book [10]. It has also been proved recently that the set of crossing-free subgraphs can be realized as a polytope [17].

A fundamental contribution was given by Ajtaj et al. [9]: the number of plane graphs on top of any $S \in \mathcal{S}_n$ is bounded from above by some fixed exponential c^n; the bound $c \leq 10^{13}$ was given there and has been successively improved up to $c \leq 472$. It is worth mentioning that a main tool developed in [9] is the nowadays famous "Crossing Lemma": every planar drawing of a graph with n vertices and $m > 4n$ edges contains at least cm^3/n^2 crossings, for some constant c. This result, independently proved by Leighton [16], has found later many applications.

In fact, the motivation in [9] was to provide an upper bound *for the number of polygonizations* (crossing-free spanning cycles) on top of any $S \in \mathcal{S}_n$. Obviously the bound for generic plane graphs applies, yet better specific bounds have been obtained for polygonizations as well as for plane triangulations, perfect matchings, spanning trees and many other classes of plane graphs; precise references are given later in this paper.

To describe the asymptotic growth of the number of graphs we use the $\mathcal{O}^*()$-, $\Omega^*()$-, and $\Theta^*()$-notation. In these notations we neglect polynomial factors and just give the dominating exponential term. Moreover when the base of the exponent is explicitly given as a numerical value this has to be seen as an approximation up to the given precision.

Maximal plane graphs, i.e., triangulations, are a case of special interest, because any plane graph can be completed to a triangulation and hence any upper bound $\mathcal{O}^*(\alpha^n)$ on the number of triangulations implies a corresponding upper bound $\mathcal{O}^*(2^{3n}\alpha^n) = \mathcal{O}^*((8\alpha)^n)$ on the number of generic plane graphs, because every triangulation has at most $3n-6$ edges and therefore contains at most 2^{3n} subgraphs. The current best upper bound for triangulations is $\mathcal{O}^*(59^n)$ and was obtained by Santos and Seidel in [20]; the aforementioned bound of $\mathcal{O}^*(472^n)$ for plane graphs is derived from that.

On the opposite direction, it is also known that every $S \in \mathcal{S}_n$ admits at least $\Omega^*(2.33^n)$ triangulations, and it has been conjectured that the number of triangulations is minimized when S is the point set called *double circle*, that has $\Theta^*(\sqrt{12}^n)$ triangulations [5].

In this paper we obtain new lower and upper bounds for the maximum and minimum, respectively, number of plane geometric graphs of different types. All given bounds will be exponential bounds of the form α^n where the goal is to optimize the base α.

More precisely, in Section 2 we prove that the number of plane graphs of several classes is minimized by point sets in convex position, a fact that was known for perfect matchings, spanning trees and spanning paths [14, 21]. Here we provide a unified approach that encompasses those results and extends to many more classes.

In Sections 3 and 4 we turn our attention to upper bounds and, in particular, we prove the existence of a certain point set that has $\Theta^*(\sqrt{72}^n) = \Omega^*(8.4853^n)$ triangulations and $\Theta^*(41.1889^n)$ plane graphs, improving the previously known best maximizing examples and disproving the common belief that the tight upper bound for the number of triangulations would be $\Theta^*(8^n)$.

We use the following notation for the indicated classes of plane graphs on top of $S \in \mathcal{S}_n$: $\mathsf{sc}(S)$: spanning cycles (Hamiltonian cycles, polygonizations); $\mathsf{pm}(S)$: perfect matchings; $\mathsf{sp}(S)$: spanning paths (Hamiltonian paths); $\mathsf{tr}(S)$: triangulations; $\mathsf{ppt}(S)$: pointed pseudo triangulations; $\mathsf{pt}(S)$: pseudo triangulations; $\mathsf{st}(S)$: spanning trees; $\mathsf{cf}(S)$: cycle-free plane graphs (forests); $\mathsf{cg}(S)$: connected graphs; $\mathsf{pg}(S)$: all plane graphs. We will use the notation $\mathsf{sc}(n)$ (similar for the other classes) if a given property holds for any point set of cardinality n.

2 Convexity Minimizes

In the following subsections we provide injective mappings of all plane graphs of Γ_n to any set of $\mathcal{S}_n$ such that the number of edges is retained.

2.1 Injective Mappings

Consider the set of all plane graphs $\mathsf{pg}(\Gamma_n)$ on top of Γ_n, an arbitrary set $S \in \mathcal{S}_n$ together with its set of plane graphs. We will show that we can map $G \in \mathsf{pg}(\Gamma_n)$ to a graph $G' \in \mathsf{pg}(S)$ in an injective way such that the number of edges of G and G' is the same. We will provide different mappings to utilize the special properties of connected or cycle-free graphs.

2.1.1 Mapping for Plane Graphs

Let G, Γ_n, and S be given as defined above. We first fix root vertices $r \in \Gamma_n$ and $r' \in S$. The root vertices have to be chosen as an arbitrary, but unique extreme vertex[1]. We then label the remaining vertices of Γ_n in clockwise order around r and the remaining vertices of S around r', respectively, see Figure 1. Consider the (possibly empty) fan of all vertices of Γ_n connected to r in G and connect the vertices with corresponding labels in S to r' in G'. By extending the inserted edges to rays as indicated in the right part of Figure 1 we are left with subsets of equal number of vertices for both, Γ_n and S. We proceed on these subsets in a recursive manner, using the just connected vertices as new, 'local' root vertices. Note that the extension of edges to rays is limited to the interior of each subset. Each local root conquers the set of all vertices in the wedge to its left, including the left-neighbored local root if it exists. The last vertex added to r (in clockwise order) plays a double-role, as it is the local root for its left and right wedge, respectively.

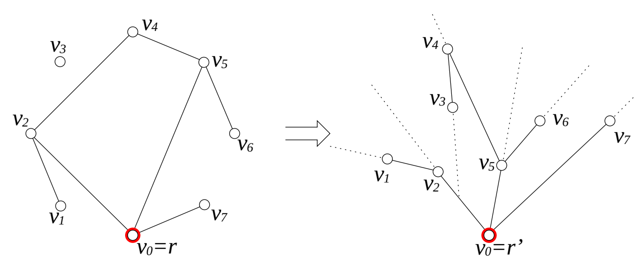

Figure 1: Injective mapping of a plane geometric graph G of Γ_n.

Note that in each recursive step the vertices of a subset are locally relabeled in both sets according to their clockwise order around the local root vertex. For example in a second recursive step in Figure 1 v_4 is the local root of the set $\{v_2, v_3, v_4\}$. As the vertices are locally resorted around v_4 the edge v_2v_4 of G maps to the edge v_3v_4 of G'. This shows why the given mapping does not guarantee isomorphic graphs. Another, simpler example is given in Figure 2. See Subsection 2.1.3 for isomorphic mappings for cycle-free plane graphs.

If a (local) root vertex is not connected to any interior vertex of the subset, in the next recursive step a new root vertex is chosen similar to the first step. For example in Figure 1 v_7 is neither

[1] For example we can choose the vertex with the smallest y-coordinate, and in case there are ties, the one with the minimum x-coordinate among them. Note that we use this method in all given figures.

connected to v_5 nor v_6, such that in the next step v_6 in Γ_n and v_5 in S are the corresponding local root vertices of the subset $\{v_5, v_6\}$ and the edge v_5v_6 is inserted in G'.

The recursion stops if the local root vertex is the only vertex of a subset. As each subset has a strictly smaller cardinality than the previous set (the previous root vertex can never be part of a subset) the process terminates.

It is important to observe that all recursive steps are independent in the sense that no edges constructed in G' cross the rays separating the subsets of S. Thus the image of G is in fact a plane graph G'. Furthermore, the injectivity of our mapping follows then from this independency and the fact that each root vertex under consideration is connected to a uniquely determined subset of vertices. This leads to the following theorem.

Theorem 1 *For any fixed number k, $0 \leq k \leq 2n-3$, the number of plane geometric graphs with k edges on top of a set of n points is minimized for sets in convex position.*

Corollary 2 *The number of plane geometric graphs on top of a set of n points is minimized for sets in convex position.*

A more formal proof of Theorem 1 works by induction over the number of points. The root vertex is chosen in a unique way and splits the problem into smaller subproblems. Note that this is still true if the chosen root of Γ_n is not connected to other vertices, as the cardinality of the remaining problem is at least reduced by one. Thus we can apply the induction hypothesis, that is, there exists an injective mapping for each subproblem. These sub-mappings are combined in a unique way via the root vertex, proving the theorem. We leave the details to the reader.

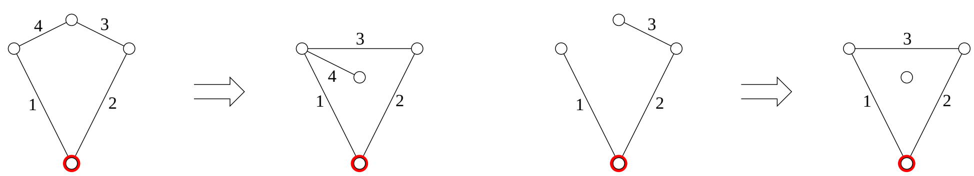

Figure 2: Transformations resulting in non-isomorphic graphs. Corresponding edges are labeled in order of their consideration.

The left side of Figure 2 shows a connected plane graph with a cycle which is transformed into a non-isomorphic graph. Thus our transformation is not suited for degree preservation, bipartite graphs etc. Also connectivity is not preserved by our mapping as can be seen from the right part of Figure 2. In the next section we will give a variation of the mapping which preserves connectivity and in Subsection 2.1.3 we will extend this to an isomorphic mapping for cycle-free plane graphs.

2.1.2 Mapping for Connected Plane Graphs

In this section we consider $\mathsf{cg}(\Gamma_n)$, the set of connected plane graphs on top of Γ_n. Let $G \in \mathsf{cg}(\Gamma_n)$.

From the right part of Figure 2 it can be seen that troubles with connectivity occur when, within the same fan, two neighbored local root vertices r_1 and r_2 of S get connected in G'. This is caused by the resorting of the subset V of vertices between r_1 and r_2 (including r_1) around r_2.

Observe that if there would have already been a connection between r_1 and r_2 in V (not necessarily a direct edge) then connectivity would not have changed. To solve this problem we allow an edge between r_1 and r_2 in G' only if they are connected in V. Note that this only preserves connectedness for connected graphs G, but does not guarantee isomorphism.

For a subset $\tilde{S}$ of Γ_n and a graph $\tilde{G}$ of $\tilde{S}$ we call a straight line through a vertex of $\tilde{S}$ which does not intesect any edge of $\tilde{G}$ a linear separator for $\tilde{S}$. For our new mapping we insert a linear

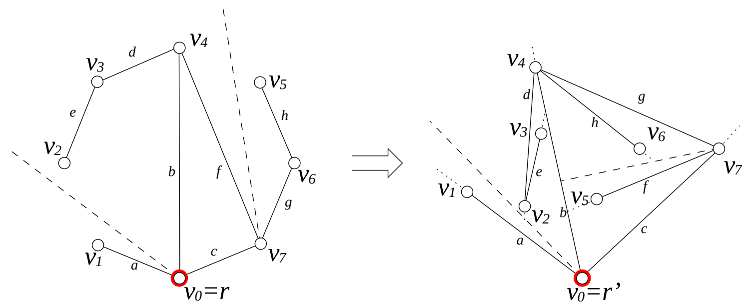

Figure 3: Using linear separators (dashed) to preserve connectedness. Corresponding edges are labeled with the same letter.

separator into a wedge of G formed by two neighbored root vertices r_1 and r_2 of a fan whenever r_1 and r_2 are not connected within the wedge, see Figure 3. In this case the subset of vertices is split into two independent parts and we have two separated recursive steps for the wedge, one with local root r_1, and one with local root r_2, respectively. Thus no edge inserted in G' crosses the linear separator.

Observe that in the example of Figure 3 vertices v_3 and v_6 would be singletons in G' using the mapping of Section 2.1.1, that is, without using linear separators.

That our new mapping indeed respects connectedness can be seen from the fact that each (local) root vertex is properly connected to its subset, and not to a neighbored root vertex of the same fan. Thus the claim follows from the recursive approach by induction.

Note that connectivity plays a crucial role when using linear separators. We can guarantee injectivity of the mapping only if, in the case that there is no linear separator, all vertices of the subset V are connected within V. See the right part of Figure 4 for a non-connected graph, where no linear separator exists for the first step but still the subset V is not a connected graph. This example shows that we can use this approach only for connected graphs.

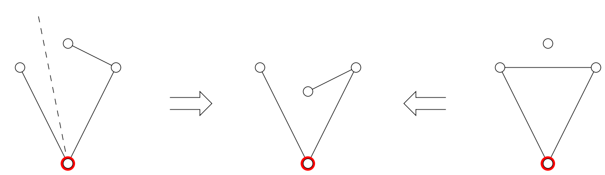

Figure 4: Using linear separators in combination with non-connected graphs causes loss of injectivity.

Theorem 3 *For any fixed number k, $0 \leq k \leq 2n-3$, the number of connected plane geometric graphs with k edges on top of a set of n points is minimized for sets in convex position.*

Corollary 4 *The number of connected plane geometric graphs on top of a set of n points is minimized for sets in convex position.*

2.1.3 Isomorphic Mapping for Cycle-Free Plane Graphs

Recall that with $\mathsf{cf}(\Gamma_n)$ we denote the set of all cycle-free, plane graphs on top of Γ_n. Let $G \in \mathsf{cf}(\Gamma_n)$ and let us point out that G is not necessarily connected.

From Figure 4 it can be seen that the mapping of the last section is non-isomorphic if cycles of G are broken or cycles in G' are closed. Therefore we now extend the mapping of the previous section. As G has been a connected graph in the last section, we had at most one linear separator for each wedge. Now, as G is cycle-free but possibly not connected, we are going to use multiple linear separators per wedge, see Figure 5.

Type	per triangulation	Type	per triangulation
$\mathsf{sc}(n)$	$\mathcal{O}^*(3^n)$ [21]	$\mathsf{pt}(n)$	$\mathcal{O}^*(3^n)$ [19]
$\mathsf{pm}(n)$	$\mathcal{O}^*(\sqrt{3}^n)$ [*]	$\mathsf{st}(n)$	$\mathcal{O}^*(5.\dot{3}^n)$ [18]
$\mathsf{sp}(n)$	$\mathcal{O}^*(3^n)$ [21]	$\mathsf{cf}(n)$	$\mathcal{O}^*(6.75^n)$ [*]
$\mathsf{tr}(n)$	1	$\mathsf{cg}(n)$	$\mathcal{O}^*(8^n)$
$\mathsf{ppt}(n)$	$\mathcal{O}^*(3^n)$ [19]	$\mathsf{pg}(n)$	$\mathcal{O}^*(8^n)$ [14]

Table 1: Number of different types of graphs per triangulation.

As G is cycle-free there is always at least one linear separator per wedge. Thus, situations like in Figure 4 cannot occur. Moreover, any recursive subset is independent from other subsets not only in the sense that no other edges cross, but also that there are no other incident edges from outside V except the ones from the (local) root. Thus, the mapping indeed generates isomorphic graphs.

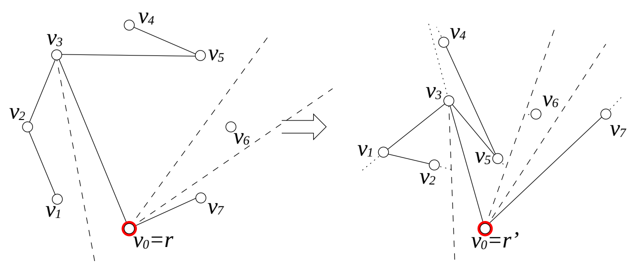

Figure 5: Cycle-free graphs: Using multiple linear separators guarantees isomorphic mapping.

The local labeling in recursive steps might still change. Although we get an isomorphic graph G', the final labeling of S needs not to be the one obtained by sorting the vertices around r'. See, for example, v_1 and v_2 in Figure 5.

Theorem 5 *There exists a mapping of all cycle-free plane geometric graphs on top of a set of n points in convex position to isomorphic plane geometric graphs on top of any set of n points.*

Corollary 6 *The number of (plane) spanning trees, spanning paths, perfect matchings, and cycle-free graphs (forests) is minimized for point sets in convex position.*

Note that for graphs with at least $k \geq 3$ edges the minimum stated in the presented theorems and corollaries is unique. That is, for any set S of n points, $S \neq \Gamma_n$, there exist strictly more such graphs than on top of Γ_n.

That the number of crossing-free perfect matchings and spanning trees is minimized for convex sets has been shown in [14]. A similar result has been obtained for the number of spanning paths [21].

With the exception of triangulations the cardinality of all classes of plane graphs considered in this paper is minimized for point sets in convex position. The accurate statement for triangulations would be that the number of plane graphs with $k = 2n - 3$ edges is minimized for sets in convex position.

The second column of Table 3 in Section 4 shows the asymptotic growth of several types of plane graphs for the convex set Γ_n. That is, except for triangulations, their asymptotically lower bounds.

3 On Upper Bounds

In order to show the bound of $\mathcal{O}^*(472^n)$ for the number of plane graphs in [14] the upper bound on the number of triangulations has been used. As any triangulation of S has at most $3n-6$ edges it contains at most 2^{3n-6} plane subgraphs. Therefore we get the bound $|\mathsf{pg}(n)| \leq 2^{3n-6}|\mathsf{tr}(n)| = \mathcal{O}^*(472^n)$. The last equality comes from the currently best upper bound for $|\mathsf{tr}(n)|$ of $\mathcal{O}^*(59^n)$ given in [20].

Type	Lower Bound	Number for Γ_{10}	Upper Bound	
$\mathsf{sc}(n)$	1	1	$\Omega^*(4.64^n)$ [14]	$\mathcal{O}^*(94^n)$ [21]
$\mathsf{pm}(n)$	$\Theta^*(2^n)$ [14]	42	$\Omega^*(3^n)$ [14]	$\mathcal{O}^*(10.04^n)$ [21]
$\mathsf{sp}(n)$	$\Theta^*(2^n)$	1 280	$\Omega^*(4.64^n)$ [14]	$\mathcal{O}^*(100.81^n)$ [21]
$\mathsf{tr}(n)$	$\Omega^*(2.33^n)$ $\mathcal{O}^*(3.47^n)$ [5]	250	$\Omega^*(8.48^n)$ [*]	$\mathcal{O}^*(59^n)$ [20]
$\mathsf{ppt}(n)$	$\Theta^*(4^n)$ [4]	1 430	$\Omega^*(12^n)$ [8]	$\mathcal{O}^*(177^n)$ [19]
$\mathsf{pt}(n)$	$\Theta^*(4^n)$ [4]	1 430	$\Omega^*(20^n)$ [8]	$\mathcal{O}^*(177^n)$ [19]
$\mathsf{st}(n)$	$\Theta^*(6.75^n)$ [13]	246 675	$\Omega^*(10.42^n)$ [11]	$\mathcal{O}^*(314.7^n)$ [18]
$\mathsf{cf}(n)$	$\Theta^*(8.22^n)$ [13]	2 117 283	$\Omega^*(11.09^n)$ [14]	$\mathcal{O}^*(398.25^n)$ [*]
$\mathsf{cg}(n)$	$\Theta^*(10.39^n)$ [13]	5 616 182	$\Omega^*(35.49^n)$ [*]	$\mathcal{O}^*(472^n)$ [14]
$\mathsf{pg}(n)$	$\Theta^*(11.65^n)$ [13]	21 292 032	$\Omega^*(41.18^n)$ [*]	$\mathcal{O}^*(472^n)$ [14]

Table 2: Asymptotic bounds for various classes of plane graphs. For easier comparison all expressions are given by their numerical values, see the related references for exact formulas ([*] stands for the paper at hand). All types except triangulations are minimized for sets in convex position.

For the maximum number of cycle-free graphs in a triangulation we get an upper bound of $\mathcal{O}(\binom{3n-6}{n-1}) = \mathcal{O}^*(6.75^n)$. For the special case of spanning trees this has recently been improved to $\mathcal{O}^*(5.\dot{3}^n)$ by observing that the dual tree has bounded degree 3 [18].

Taking the average vertex degree in a triangulation into account, a bound of $\mathcal{O}^*(3^n)$ for the number of spanning paths in a given triangulation can be shown [21]: Staring at an arbitrary beginning vertex construct the path edge-wise step by step. At each vertex the number of possible ways to continue, that is, to choose the next edge, is its effective degree. Here effective degree means that we do not count edges of the triangulation which have already been considered before. The reason is that once a vertex v has been visited no edge incident to v can be used later on to continue the path (even if the edge has not been chosen, as v cannot be visited again). Hence, every edge is considered only once and therefore the sum of the effective degree of all vertices is the number of edges of the triangulation, bounded by $3n - 6$. Essentially the number of different spanning paths is the product of the effective degrees. This product is maximized by uniformly distributing the over-all degree, and the given bound follows.

Using the same arguments we get a bound of $\mathcal{O}^*(\sqrt{3}^n)$ for the number of perfect matchings in a triangulation. Here a single step consists of choosing the leftmost unused vertex and matching it to one of its (effective) incident neighbours.

The bound on $|\mathsf{sp}(n)|$ also implies an upper bound for the number of spanning cycles: For every spanning cycle $C \in \mathsf{sc}(S)$, $S \in \mathcal{S}_n$, we get n spanning paths by omitting one edge of C. As any two elements of $\mathsf{sc}(S)$ differ by at least two edges this implies $|\mathsf{sp}(S)| \geq n \cdot |\mathsf{sc}(S)|$.

Most of the upper bound constructions for various classes of plane graphs are based on the just mentioned relation to triangulations. For example the bound on the number of cycle-free graphs in a triangulation implies the upper bound of $\mathcal{O}^*(398.25^n)$ for the number of cycle-free graphs given in Table 2. Only recently different approaches have been investigated, an outstanding result being the $\mathcal{O}^*(10.04^n)$ bound for crossing-free perfect matchings [21]. Syntactically their proof follows the proof of [20] bounding the number of triangulations, but uses novel ideas and refined observations.

Interestingly the result for crossing-free perfect matchings can be used to bound the number of spanning cycles and spanning paths [21]: Observe that for even n every spanning cycle contains two crossing-free perfect matchings. Thus $|\mathsf{sc}(n)| \leq |\mathsf{pm}(n)|^2$. Similarly every spanning path contains a crossing-free perfect matching on n points and a crossing free perfect matching on $n - 2$ points (omit start and endpoint). We thus get for both structures an upper bound of $\mathcal{O}^*(100.81^n)$. Most recently this bound has been improved to $\mathcal{O}^*(94^n)$ for spanning cycles [21].

Type	Convex Set	Double Circle	Double Chain	double zig-zag chain
$\mathsf{sc}(n)$	1	$\mathcal{O}^*(4.83^n)$ [*]	$\Omega^*(4.64^n)$ $\mathcal{O}^*(5.61^n)$ [14]	
$\mathsf{pm}(n)$	$\Theta^*(2^n)$ [14]	$\Theta^*(2.20^n)$ [*]	$\Theta^*(3^n)$ [14]	
$\mathsf{sp}(n)$	$\Theta^*(2^n)$	$\mathcal{O}^*(4.83^n)$ [*]	$\Omega^*(4.64^n)$ [14]	
$\mathsf{tr}(n)$	$\Theta^*(4^n)$	$\Theta^*(\sqrt{12}^n)$ [5, 8]	$\Theta^*(8^n)$ [14]	$\Theta^*(8.48^n)$ [*]
$\mathsf{ppt}(n)$	$\Theta^*(4^n)$	$\Theta^*(\sqrt{28}^n)$ [8]	$\Theta^*(12^n)$ [8]	
$\mathsf{pt}(n)$	$\Theta^*(4^n)$	$\Theta^*(\sqrt{40}^n)$ [8]	$\Theta^*(20^n)$ [8]	
$\mathsf{st}(n)$	$\Theta^*(6.75^n)$ [13]	$\Omega^*(6.89^n)$	$\Omega^*(10.42^n)$ [11]	
$\mathsf{cf}(n)$	$\Theta^*(8.22^n)$ [13]	$\Omega^*(8.22^n)$	$\Omega^*(11.09^n)$ [14]	
$\mathsf{cg}(n)$	$\Theta^*(10.39^n)$ [13]	$\Omega^*(10.84^n)$	$\Omega^*(35.49^n)$ [*]	$\Omega^*(32.49^n)$ [*]
$\mathsf{pg}(n)$	$\Theta^*(11.65^n)$ [13]	$\Theta^*(15.0046^n)$ [*]	$\Theta^*(39.80^n)$ [14][*]	$\Theta^*(41.19^n)$ [*]

Table 3: Special configurations and their asymptotic number of graphs.

4 Special Configurations

In the following subsections we consider two well known configurations, namely the double circle [5] and the double chain [14], as well as a new configuration, the so-called double zig-zag chain (DZZC). We will show that the DZZC is a new extremal example for maximizing the asymptotic number of triangulations and of all plane graphs on top of a given point set.

4.1 Double Circle and Double Chain

For completeness we include the following results on double circles and double chains. All proofs and arguments are omitted for this extended abstract. Appendix A contains the full version of this section, including all proofs.

Theorem 7 *Let $|\mathsf{pg}(DC_n)|$ be the number of plane geometric graphs of the double circle containing n points. Then* $|\mathsf{pg}(DC_n)| = \Theta^*\left(\left(4\sqrt{7+5\sqrt{2}}\right)^n\right) = \Theta^*(15.0045^n)$.

Theorem 8 *The double chain has $\Omega^*(35.49^n)$ connected plane graphs.*

Theorem 9 *The double circle of n points has $\Theta^*(2.197368^n)$ crossing-free perfect matchings.*

With similar arguments as used at the end of section 3 we therefore obtain

Corollary 10 *The double circle of n points has $\mathcal{O}^*(4.828427^n)$ plane spanning cycles and spanning paths.*

With a similar approach we can obtain lower bounds for the number of spanning trees and connected graphs for the double circle which are slightly better than the bounds for the convex case, cf. Table 3. We do not expect these bounds to be tight.

4.2 The Double Zig-Zag Chain - a new Extremal Configuration

In this section we combine the double circle and the double chain in order to obtain a new extremal configuration, the so-called double zig-zag chain (DZZC).

While the double circle has $\Theta^*(\sqrt{12}^n)$ triangulations and is thus conjectured to minimize this number [5], the double chain was up to now the configuration with the asymptotically highest number of triangulations, namely $\Theta^*(8^n)$ [14]. It was widely believed (including most of the authors) that this could be the true upper bound for the number of triangulations.

To obtain the double zig-zag chain take two distorted double circles with $\frac{n}{2}$ points each and combine them within a convex quadrilateral as indicated in Figure 6. The example is similar to the double chain, but instead of two concave chains we now have two zig-zag chains of edges splitting the area of the quadrilateral into three parts. These edges are unavoidable in the sense that they are not crossed by any other edge spanned by the point set. For example the edges of the zig-zag chains will have to be part of any triangulation. Note that the two zig-zag configurations are at sufficient distance from each other such that any vertex of a chain can 'see' all vertices of the opposite chain, that is, an edge connecting a vertex of the lower chain to a vertex of the upper chain does not cross a zig-zag edge. Both, upper and lower part, are double circles with $\frac{n}{2}$ vertices and thus $\Theta^*(\sqrt{12}^{n/2})$ triangulations each. To be precisely there is one interior vertex near the horizontal convex-hull edges missing in each subset, which does of course not influence any asymptotic counting arguments.

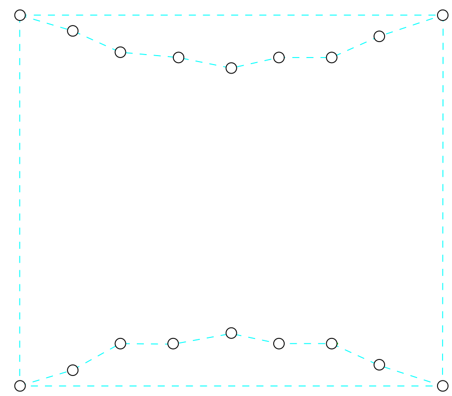

Figure 6: Double zig-zag chain: combining double circle and double chain.

4.2.1 The number of plane graphs of the DZZC

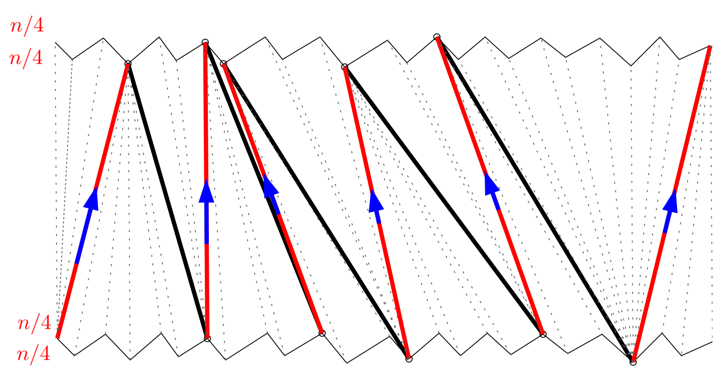

Figure 7: Counting plane graphs in the interior of the double zig-zag chain.

We first count the number of plane geometric graphs in the interior of the double zig-zag chain. To this end we are going to choose m points, $0 \leq m \leq \frac{n}{2}$, from each zig-zag chain. The points of each zig-zag chain can be viewed as lying on a smaller and a larger circle. For the upper zig-zag chain we choose i_1 points from the larger and j_1 points from the smaller circle, $i_1 + j_1 = m$. Similarly we choose i_2 and j_2 points from the lower zig-zag chain.

There are $\sum_{0 \leq m \leq \frac{n}{2}} \sum_{i_1,i_2,i_1+j_1=i_2+j_2=m} \binom{\frac{n}{4}}{i_1} \cdot \binom{\frac{n}{4}}{j_1} \cdot \binom{\frac{n}{4}}{i_2} \cdot \binom{\frac{n}{4}}{j_2}$ ways to do so. Then we connect the chosen points by pairing a point from the upper zig-zag chain with a point from the lower zig-zag chain by scanning them from left to right. These pairs are indicated as black edges in Figure 7. Next we draw red edges connecting the black edges in a way that we connect the lower endpoint of the 'left' black edge to the upper endpoint of the 'right' black edge. If the starting (ending) point of the lower (upper) zig-zag chain is not used by a black edge, then the first (last) red edge uses this point instead of an endpoint of a black edge.

We complete the drawing to a triangulation of the interior by connecting the two zig-zag chains in a fan like manner to the endpoints of the red and black edges. Let us color these edges gray (dotted edges in Figure 7). Note that we can flip some of the gray edges (connected to the smaller circle of a zig-zag chain) to obtain different triangulations, still containing the red and black edges. Here a flip exchanges the two diagonals of the convex quadrilateral formed by the two adjacent triangles.

Next we consider all subgraphs of the obtained triangulations which contain the black segments. For a flippable gray edge we have three possibilities (draw it, delete it, flip it), for a non-flippable gray edge we have two possibilities. We also have two possibilities for the red edges (draw it or not). Note that we do not flip red edges for the following reason. It is important to observe that any obtained subgraph is uniquely assigned to its triangulation, that is, to the given set of black edges. In other words for a given subgraph its triangulation can be uniquely restored: We can always detect (and insert) the red and black edges: By starting from the leftmost point of the lower zig-zag chain we draw an edge (or simply detect whether it is already there) to the rightmost visible point on the upper zig-zag chain. This gives us the first red edge and the starting point of the next black edge. The black edge is determined as the rightmost edge incident to this vertex and going back to the lower zig-zag chain. Continuing in the same manner this gives us the remaining red and black edges. Note that the above argumentation still goes through when flippable gray edges have been flipped arbitrarily.

We now count the number of the constructed subgraphs. There are $\frac{n}{2} - (j_1 + j_2)$ flippable gray edges (one for each non-chosen point of the two smaller circles). Analogously, there are $\frac{n}{2} - (i_1 + i_2)$ non-flippable gray edges. And there are m black and $m + \{-1, 0, +1\}$ red edges.

Thus, we obtain the number of plane geometric graphs of the interior of the double zig-zag chain by multiplying all factors as $\sum_{0 \le m \le \frac{n}{2}} \sum_{i_1, i_2, i_1 + j_1 = i_2 + j_2 = m} \binom{\frac{n}{4}}{i_1} \cdot \binom{\frac{n}{4}}{j_1} \cdot \binom{\frac{n}{4}}{i_2} \cdot \binom{\frac{n}{4}}{j_2} \cdot 3^{\frac{n}{2} - (j_1 + j_2)} \cdot 2^{\frac{n}{2} - (i_1 + i_2) + m + \{-1, 0, +1\}}$.

Neglecting polynomial factors the asymptotic of this sum is determined by its largest element, since we have only a linear number of terms. As $i_1 + j_1$ is independent from $i_2 + j_2$ the maximum is obtained for $i_1 = i_2$ and $j_1 = j_2$ by symmetry.

Using Stirling's formula $n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ one derives the well known estimate $\binom{n}{\alpha n} = \Theta(n^{-\frac{1}{2}} 2^{H(\alpha) n})$ where $H(\alpha) = -(\alpha \log_2 \alpha + (1 - \alpha) \log_2 (1 - \alpha))$ denotes the binary entropy function. From this relation we get $i = (\sqrt{2} - 1)\frac{n}{4}$ and $j = \frac{\sqrt{2}}{3 + \sqrt{2}} \frac{n}{4}$ which evaluates to $\Theta^*(3,88215^n)$ plane graphs for the inner part of the double zig-zag chain.

Together with Theorem 7 we obtain

Theorem 11 *The double zig-zag chain of n points contains $\Theta^*(41.1889^n)$ plane geometric graphs.*

In [14] a lower bound of $\Omega^*(39.80^n)$ plane graphs of the double chain has been shown. Using the argumentation from above we will strengthen this to $\Theta^*(39.80^n)$, which implies that the double zig-zag chain has in fact asymptotically more plane graphs than the double chain. The main difference to the above approach is that for a chain of the double chain we do not need to distinguish between points of the smaller and larger circles, but choose all vertices from the same chain.

In addition, as none of the edges in the inner part is flippable any more, red and gray edges provide now a factor of 2. Thus if we choose $c \cdot \frac{n}{2}$ points of each chain for the black edges, the general term of the sum simplifies to $\binom{\frac{n}{2}}{c\frac{n}{2}}^2 \cdot 2^{n - c\frac{n}{2}}$. This term is maximized for $c = \sqrt{2} - 1$ leading to $\Theta^*(39.79898^n)$ plane geometric graphs of the double chain, the exact base being $20 + 14\sqrt{2}$.

4.2.2 The number of triangulations of the DZZC

To count the number of triangulations for the central part of the double zig-zag chain we use the same approach as in Section 4.2.1. As every constructed edge has to be in the triangulations, we only get a factor 2 for gray, flippable edges. Non-flippable edges and red edges do not provide a multiplicative factor. We thus get $\sum_{0 \leq m \leq \frac{n}{2}} \sum_{i_1, i_2, i_1+j_1=i_2+j_2=m} \binom{\frac{n}{4}}{i_1} \cdot \binom{\frac{n}{4}}{j_1} \cdot \binom{\frac{n}{4}}{i_2} \cdot \binom{\frac{n}{4}}{j_2} \cdot 2^{\frac{n}{2}-(j_1+j_2)}$ triangulations. Again for symmetry reasons we assume $i_1 = i_2$ and $j_1 = j_2$ for the maximal term of the sum. Moreover set $i = c \cdot \frac{n}{4}$ and $j = d \cdot \frac{n}{4}$ and we have to maximize the term $\binom{\frac{n}{4}}{c \cdot \frac{n}{4}}^2 \cdot \binom{\frac{n}{4}}{d \cdot \frac{n}{4}}^2 \cdot 2^{(1-d)\frac{n}{2}}$ for $0 \leq c, d \leq 1$.

Again we make use of the binary entropy function as described in Section 4.2.1 and conclude that $c = \frac{1}{2}$ gives the maximum for the first factor, namely $\Theta^*(2^{\frac{n}{2}})$. For d we have to maximize $2^{H(d)\frac{n}{2}+(1-d)\frac{n}{2}}$ which is equivalent to maximizing $H(d) - d$. It is well known that the maximum is obtained for $d = \frac{1}{3}$, resulting in $\Theta^*(2^{\frac{\log_2 3}{2} n})$. Combining both factors gives $\Theta^*(2^{H(1/3)n/2+5n/6})$ triangulations, which can be simplified to $\Theta^*(\sqrt{6}^n) = \Theta^*(2.44949^n)$ triangulations for the central part.

In total we thus get $\Theta^*(\sqrt{12}^{\frac{n}{2}})^2 \cdot \Theta^*(\sqrt{6}^n) = \Theta^*((6\sqrt{2})^n) = \Theta^*(8.48528^n)$ triangulations for the double zig-zag chain.

Theorem 12 *The double zig-zag chain of n points contains $\Theta^*(8.48528^n)$ triangulations.*

5 Further Work and Open Problems

The most challenging question is of course to close the gap between maximizing examples and upper bounds. Here Sharir and Welzl [21] recently have made enormous progress on the upper bounds.

Obviously further work is required to improve (or even fill in) the entries in Tables 1 to 3. In Tables 1 and 2 the goal is to close or at least narrow the gaps between lower and upper bounds, while for Table 3 several entries, especially for the double zig-zag chain, are missing.

Concerning our lower bound construction an interesting question is the following: Does there exists an example that shows that an isomorphic mapping from any planar graph on top of a convex point set to any other point set (of the same cardinality) is not possible? If such an example does not exist, can we find a unified isomorphic mapping that works for all graphs?

6 Acknowledgments

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A Double Circle and Double Chain

This appendix contains the complete version of Section 4.1.

A.1 The number of plane graphs of the Double Circle

Theorem 13 *Let $|\mathsf{pg}(DC_n)|$ be the number of plane geometric graphs of the double circle containing n points. Then $|\mathsf{pg}(DC_n)| = \sum_{i=0}^{\frac{n}{2}} (-1)^i \cdot \binom{\frac{n}{2}}{i} \cdot 2^{i+\frac{n}{2}} \cdot |\mathsf{pg}(\Gamma_{n-i})|$, where $|\mathsf{pg}(\Gamma_m)|$ is the number of plane geometric graphs of the convex m-gon.*

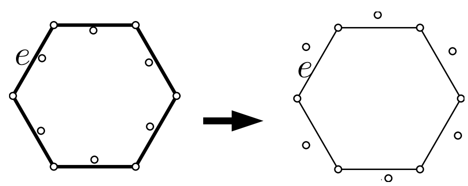

Figure 8: Moving the interior points to the outside of the convex hull.

Proof. We move every interior point of the double circle to the outside of its corresponding convex hull edge e, see Figure 8. A plane geometric graph of the double circle might have crossings in the resulting drawing. We count the number of plane geometric graphs of this point set which do not contain any of the former convex hull edges (i.e., an edge e in the right drawing of Figure 8). Since a convex hull edge of the double circle appears in exactly half of all plane graphs, we simply have to multiply this number by $2^{\frac{n}{2}}$.

If edge e belongs to a plane graph, then there are four possibilities to connect (or not) the point which is separated by e to the plane graph. Hence, we assume this separated point does not belong to the graph. But then, e is a convex hull edge and appears in half of all graphs. Since our new point set is convex, the edge e appears in $\frac{|\mathsf{pg}(\Gamma_{n-1}))|}{2} \cdot 4$ plane graphs of $\mathsf{pg}(\Gamma_n)$. Similarly, i edges 'of type e' appear simultaneously in $\frac{|\mathsf{pg}(\Gamma_{n-i})|}{2^i} \cdot 4^i$ plane graphs of $\mathsf{pg}(\Gamma_n)$. Now, in order to count all plane geometric graphs not containing an edge 'of type e' we can use the inclusion-exclusion principle. We obtain $|\mathsf{pg}(DC_n)| = \Big(|\mathsf{pg}(\Gamma_n)| - \frac{n}{2} \cdot 2 \cdot |\mathsf{pg}(\Gamma_{n-1})| + \binom{\frac{n}{2}}{2} \cdot 4 \cdot |\mathsf{pg}(\Gamma_{n-2})| - \cdots + (-1)^{\frac{n}{2}} \cdot 2^{\frac{n}{2}} \cdot |\mathsf{pg}(\Gamma_{\frac{n}{2}})|\Big) \cdot 2^{\frac{n}{2}}$. □

Corollary 14 (Theorem 7 in the extended abstract) $|\mathsf{pg}(DC_n)| = \Theta^*\left(\left(4\sqrt{7+5\sqrt{2}}\right)^n\right) = \Theta^*(15.0045^n)$.

Proof. We know that $|\mathsf{pg}(\Gamma_m)| = \Theta^*\left(\left(6+4\sqrt{2}\right)^m\right)$ [13] and therefore we get $|\mathsf{pg}(DC_n)| = \sum_{i=0}^{\frac{n}{2}} (-1)^i \cdot \binom{\frac{n}{2}}{i} \cdot 2^{i+\frac{n}{2}} \cdot \Theta^*\left(\left(6+4\sqrt{2}\right)^{n-i}\right) = \Theta^*\left(\sum_{i=0}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{i} \cdot \left(\frac{-2}{6+4\sqrt{2}}\right)^i\right) \cdot (6+4\sqrt{2})^n \cdot 2^{\frac{n}{2}}\right)$ Using the binomial theorem $\sum_{i=0}^{t} \binom{t}{i} \cdot (x)^i = (1+x)^t$ we get $|\mathsf{pg}(DC_n)| =$ $\Theta^*\left(\left(1-\frac{1}{3+2\sqrt{2}}\right)^{\frac{n}{2}} \cdot (6\sqrt{2}+8)^n\right)$ which simplifies to the given formula. □

A.2 The number of connected plane graphs of the Double Chain

Theorem 15 (Theorem 8 in the extended abstract) *The double chain has $\Omega^*(35.49^n)$ connected plane graphs.*

Proof. Consider all graphs of the double chain such that the subgraphs of both convex $\frac{n}{2}$-gons of the double chain are connected. Adding one vertical edge of the convex hull of the double chain gives a connected graph. From [14] and Section 4.2.1 we know that there are $\Theta^*(39.80^n)$ plane graphs in the double chain. Moreover a convex n-gon has $\Theta^*(11.65^n)$ plane graphs and $\Theta^*(10.39^n)$ connected plane graphs [13]. We thus have to correct the number 39.80^n by the factor $(\frac{10.39}{11.65})^n$ and get a lower bound of $\Omega^*(35.49^n)$ for the number of connected plane graphs of the double chain. □

A.3 The number of non-crossing matchings in the double circle

Let S_e be a set of $2m - k$ points in convex position, and let S_i be a set of k points interior to $CH(S_e)$, $k \le m$, each one infinitesimally close to a different midpoint of an edge of $CH(S_e)$. When $|S_e| = |S_i| = k = m$ the configuration is the *double circle*.

Let $\mu(2m, k)$ denote the number of non-crossing perfect matchings in $S_e \cup S_i$ for the special case in which all the points in S_i correspond to k *consecutive* edges of $CH(S_e)$.

Lemma 16 *For $2 \le k \le m-1$, $m \ge 3$ we have $\mu(2m, k) = \mu(2m, k-1) + \mu(2m-2, k-2)$.*

Proof. Let a and b be the endpoints of an edge of $CH(S_e)$ such that there is a point $c \in S_i$ close to the midpoint of ab, and one of the neighboring edges of $CH(S_e)$ has also a point from S_i but the other one does not (refer to Figure 9). Let us call C_1 this configuration of points, and let C_2 be the configuration obtained from C_1 by replacing c with a point d exterior to $CH(S_e)$ and very close to the midpoint of ab. Let C_3 be the configuration obtained from C_1 after the removal of the points a and b.

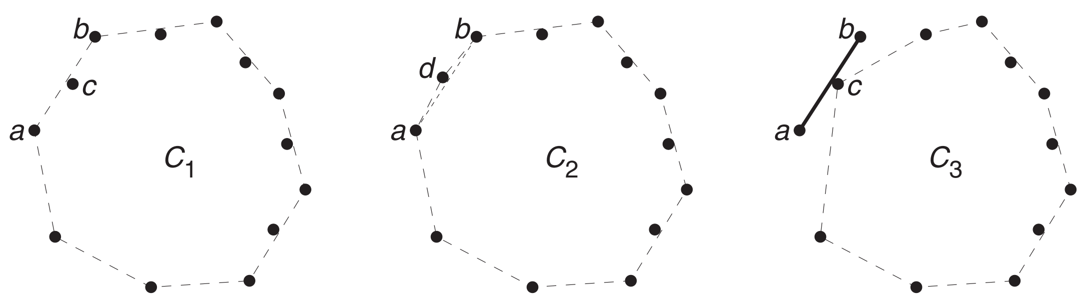

Figure 9: Illustrating the recursion for $\mu(2m, k)$.

Now notice that non-crossing matchings in C_1 are in one-to-one correspondence with the matchings in C_2, with d playing the role of c, with the exception of those matchings in which a is matched with b; these are in one-to-one correspondence with the matchings in C_3, which proves the claim. □

Lemma 17 *The following equalities hold:*

(a) $\mu(2m, 0) = C_m$;

(b) $\mu(2m, 1) = \mu(2m, 0) + \mu(2m-2, 0) = C_m + C_{m-1}$;

(c) $\mu(2m, m) = \mu(2m, m-1) + \mu(2m-2, m-3)$;

where C_m is the m-th Catalan number.

Proof. Equality (a) is well known, see for example [13]. Equality (c) and the first part of (b) are proved by moving a point from S_i to the exterior of $CH(S_e)$ as in the proof of Lemma 16. □

Proposition 18 *For* $2 \leq k \leq m-1$, $m \geq 3$ *we have* $\mu(2m,k) = \sum_{s=0}^{k+1} \binom{k+1-s}{s} C_{m-s}$.

Proof. First notice that the binomial coefficient is 0 for $k+1-s < s$. The proof is straightforward using Lemma 16 and induction. The base case $\mu(2m,2) = C_m + 2C_{m-1}$ is proved combining Lemma 16 with Lemma 17. □

Finally, combining Lemma 17 with Proposition 18 we obtain the number of perfect matchings of the double circle.

Corollary 19 *The number of non-crossing perfect matchings of the double circle is*

$$\mu(2m,m) = \sum_{i=0}^{m} \binom{m-i}{i} C_{m-i} + \sum_{j=0}^{m-2} \binom{m-2-j}{j} C_{m-1-j}.$$

The preceding result can be used in order to obtain an asymptotic estimate of $\mu(2m,m)$. The generic term in the expression is roughly $\binom{(1-\alpha)m}{\alpha m} C_{(1-\alpha)m} \approx 2^{(1-\alpha)\left[H(\frac{\alpha}{1-\alpha})+2\right]m}$, and elementary computations show that the exponent is maximized for $\alpha = (2-\sqrt{2})/4 \approx 0.1464466$, therefore the number of perfect matchings of a double circle with $n = 2m$ points is $\Theta^*(2^{2.271553m}) = \Theta^*(4.828427^m) = \Theta^*(\sqrt{4.828427}^n) = \Theta^*(2.197368^n)$.

Theorem 20 (Theorem 9 in the extended abstract) *The double circle of* n *points has* $\Theta^*(2.197368^n)$ *crossing-free perfect matchings* [2].

With similar arguments as used at the end of section 3 we obtain

Corollary 21 (Corollary 10 in the extended abstract) *The double circle of* n *points has* $\mathcal{O}^*(4.828427^n)$ *plane spanning cycles and spanning paths.*

With a similar approach we can obtain lower bounds for the number of spanning trees and connected graphs for the double circle which are slightly better than the bounds for the convex case, cf. Table 3. We do not expect these bounds to be tight.

[2] The precise base can be computed from the formula $\sqrt{2^{\frac{\log 16+\sqrt{2}(\log 4-\log 8)+(\sqrt{2}-2)\log(2-\sqrt{2})+(2+\sqrt{2})\log(2+\sqrt{2})}{\log 16}}}$.