On the chromatic number of some geometric type Kneser graphs

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July 27, 2004

Abstract

We estimate the chromatic number of graphs whose vertex set is the set of edges of a complete geometric graph on n points, and adjacency is defined in terms of geometric disjointness or geometric intersection.

1 Introduction

For integers $n > k > t > 0$, the general Kneser graph $K(n, k, t)$ is defined as the graph whose vertices are all k-subsets of the set $\{1, 2, \ldots, n\}$ and two such sets X and Y joined by an edge if and only if $|X \cap Y| < t$ (see e.g. [3]). Kneser [5] conjectured the following result on the chromatic number: $\chi(K(n, k, 1)) = n - 2k + 2$ for all $2 \leq k \leq n/2$. This was proved by Lovász [8] using tools from algebraic topology. A shorter proof was given shortly after by Bárány [1], and a purely combinatorial proof has been obtained recently by Matoušek [9].

In this paper we discuss the following two geometric versions of the problem. Let S be a set of n points in *general position* in the plane (i.e., no three points collinear), and consider two graphs $D(S)$ and $I(S)$ whose vertex set consists of all subsets of k points in S. Two such sets X and Y are adjacent in $D(S)$ if and only if their convex hulls are disjoint, and are adjacent in $I(S)$ if and only if their convex hulls intersect.

In the sequel, we restrict ourselves to the case $k = 2$, and refer to $D(S)$ and $I(S)$, respectively, as the *segment disjointness graph* of S and the *segment intersection graph* of S. Let

$$d(n) = \max\{\chi(D(S)) : S \subset \mathbb{R}^2 \text{ is in general position}, |S| = n\},$$

and similarly

$$i(n) = \max\{\chi(I(S)) : S \subset \mathbb{R}^2 \text{ is in general position}, |S| = n\}.$$

If we restrict our attention to point sets in convex position, then the corresponding functions are denoted by $d_c(n)$ and $i_c(n)$, respectively. In this case we denote the corresponding graphs $D(n)$ and $I(n)$, since clearly they depend only on the number n of points and not on the particular position of the points. With this notation $d_c(n) = \chi(D(n))$ and $i_c(n) = \chi(I(n))$. We clearly have $d_c(n) \leq d(n)$ and $i_c(n) \leq i(n)$.

Our first result provides bounds for the functions $d_c(n)$ and $d(n)$. Throughout this paper, logarithms are to the base two.

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Theorem 1 *For any $n \geq 3$ we have*

(i) $2\lfloor \frac{n+1}{3} \rfloor - 1 \leq d_c(n) \leq \min\left(n-2, n - \frac{\lfloor \log n \rfloor}{2}\right)$.

(ii) $5\lfloor \frac{n}{7} \rfloor \leq d(n) \leq \min\left(n-2, n + \frac{1}{2} - \frac{\lfloor \log \log n \rfloor}{2}\right)$.

Notice that $D(n)$ is a spanning subgraph of $K(n,2,1)$. The fact that the chromatic numbers $d_c(n)$ and $d(n)$ are smaller than the corresponding value $n-2 = \chi(K(n,2,1))$ shows a different qualitative behavior when geometry comes into play.

Our second result gives bounds for the functions $i_c(n)$ and $i(n)$.

Theorem 2 *For any $n \geq 3$ we have*

(i) $i_c(n) = n$.

(ii) $n \leq i(n) \leq Cn^{3/2}$*, for some constant $C > 0$.*

It is easy to check that for fixed $k \geq 3$, $d(n) = \Theta(n)$ and $i(n) = \Theta(n^k)$, and this is why we restrict our attention to the case $k = 2$. Let $3 \leq k \leq n/2$ be a fixed integer, and S be a set of n points in general position in the plane. That $d(n) = O(n)$ follows from the upper bound $\chi(K(n,k,1)) \leq n - 2k + 2$; see also the proof of Theorem 1 below. To see that $d(n) = \Omega(n)$, consider sweeping a vertical line across S to get $\lfloor n/k \rfloor$ subsets of k points each, whose convex hulls are pairwise disjoint, thus each requires a different color. Consider next $i(n)$. Coloring each subset of k points of S with a different color gives the trivial upper bound $i(n) = O(n^k)$. If S is a set of n points in convex position, say in clockwise order, it can be partitioned into k consecutive groups of at least $\lfloor n/k \rfloor$ points each. The set of convex k-gons formed by selecting one point in each group, consists of pairwise intersecting polygons and has size $\Omega(n^k)$. Thus each such subset of k points requires a distinct color.

There is a yet another graph which is worth exploring, suggested to us by János Pach. We say that two segments *cross* if they have an interior point in common. For a set of points S, define the graph $W(S)$ whose vertices are all the segments determined by pairs of points in S, two of them being adjacent if they do *not* cross. Define $w(n)$ as the maximum of $\chi(W(S))$ among sets S of n points in general position, and define $w_c(n)$ analogously for points in convex position. We clearly have $w_c(n) \leq w(n)$. Observe that $D(S)$ is a spanning subgraph of $W(S)$, so $d(n) \leq w(n)$.

Theorem 3 *For any $n \geq 3$ we have*

(i) $w_c(n) = \Theta(n \log n)$.

(ii) $c_1 n \log n \leq w(n) \leq c_2 n^2 \cdot \frac{\log \log n}{\log n}$*, for some constants $c_1, c_2 > 0$.*

In fact we discovered after our research that the case for $w_c(n)$ had already been studied and the same bound obtained [6], nevertheless we include our proof for the sake of completeness.

As a final clarification, let us mention what the independent sets (i.e., color classes) and cliques are for each type of coloring. Recall that a (geometric) *thrackle* is a geometric graph whose every pair of edges intersect (at an interior point or at a common endpoint). For Theorem 1, the independent sets are thrackles, and the cliques are plane matchings. For Theorem 2, the independent sets are plane matchings, and the cliques are thrackles. For Theorem 3, the independent sets are crossing matchings, and the cliques are plane graphs.

2 Proof of Theorem 1

Lower bounds. A *geometric graph* $G = (V, E)$ is a graph drawn in the plane so that the vertex set V consists of points in the plane, no three of which are collinear, and the edge set E consists of straight line segments between points of V (cf. [10]).

Theorem 4 (G. Károlyi, J. Pach and G. Tóth, [4]) *If the edges of a complete geometric graph on n vertices are colored by two colors, there exist $\lfloor \frac{n+1}{3} \rfloor$ pairwise disjoint edges of the same color.*

To prove a lower bound of $2\lfloor \frac{n+1}{3} \rfloor - 1$ on $d_c(n)$ and $d(n)$, consider a set S of n in general position in the plane. Let $C_1, C_2, \ldots, C_k$ be a k-coloring of $D(S)$. Consider the following bipartition of the segments with endpoints in S:

$$A = C_1 \cup C_2 \ldots \cup C_{\lfloor \frac{k}{2} \rfloor}, \;\; B = C_{\lfloor \frac{k}{2} \rfloor + 1} \cup \ldots \cup C_k.$$

Since either A or B must contain $\lfloor \frac{n+1}{3} \rfloor$ disjoint edges (by Theorem 4), which must in turn belong to different color classes, we get

$$\left\lceil \frac{k}{2} \right\rceil \geq \left\lfloor \frac{n+1}{3} \right\rfloor,$$

from which the bound follows.

For points in general position, we show a better lower bound of $5\lfloor \frac{n}{7} \rfloor$. First note that for any positive integers i and j, such that $n \geq i \cdot j$, we have $d(n) \geq i \cdot d(j)$, since we can use place i copies of j points such that the convex hulls of the j-sets are pairwise disjoint, and the resulting set is in general position. Set $i = \lfloor \frac{n}{7} \rfloor$ and $j = 7$, and consider the configuration of seven points shown in Fig. 1, with four points as vertices of a rectangle and three points inside the rectangle and close to the middle points of three rectangle sides.

The case analysis below shows that the chromatic number of the segment disjointness graph of the configuration in Fig. 1 is at least five, hence $d(7) = 5$. Assume for contradiction that four colors, say green, blue, red and purple, are sufficient.

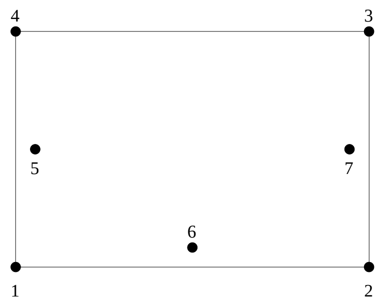

Figure 1: A configuration of seven points: proof of the lower bound on $d(n)$.

Case 1 The triangle with vertices 5, 6 and 7 is monochromatic, say green: 56, 57 and 67 are green. Color 12 with blue and 34 with red. Clearly, one of the segments 15 and 27 is blue and the other is purple. By symmetry, we can assume that 15 is blue and 27 is purple. Then 45 is red and 37 is purple. The segment 26 cannot be colored with any color, which is a contradiction.

Case 2 The triangle with vertices 5, 6 and 7 is bichromatic, having the segments 65 and 67 of the same color: say 65 and 67 are green, and 57 is red. Color 12 with blue and 34 with purple. Clearly, one of the segments 45 and 37 is red and the other is purple. By symmetry, we can assume that 45 is red. Then 27 must be colored blue, and 14 cannot be colored, contradiction.

Case 3 The triangle with vertices 5, 6 and 7 is bichromatic, having the segments 56 and 57 of the same color: say 56 and 57 are green, and 67 is red. Color 12 with blue and 34 with purple. Clearly, one of the segments 14 and 23 is purple and the other is blue. If 14 is purple (and 23 is blue), then 37 is red, and there is no color left for 16, contradiction. If 14 is blue (and 23 is purple), 45 is green, 27 is red, 26 is red, and then one cannot assign any color to 47, contradiction.

Case 4 The triangle with vertices 5, 6 and 7 is trichromatic, say red, blue and green. It is easy to see that segments 12 and 34 are disjoint and need two new colors. We have obtained again a contradiction.

By symmetry with Case 3, the case when the triangle with vertices 5, 6 and 7 is bichromatic, having the segments 75 and 76 of the same color, is omitted.

Observations. A weaker lower bound of $\frac{n-1}{2}$ on $d_c(n)$ and $d(n)$, follows immediately from the fact that any geometric graph with n vertices and at least $n+1$ edges contains two disjoint edges [2] (see also [10]).

It is likely that our lower bound on $d(n)$ can be improved using another "small" point configuration, having more points than the one in Fig. 1.

Upper bounds. Let S be a set of n points in general position in the plane and $D(S)$ its corresponding disjointness graph. Since $D(S)$ is a subgraph of $K(n,2,1)$, the upper bound of $n-2$ on $\chi(K(n,2,1))$ applies to both $d_c(n)$ and $d(n)$ as well. To be precise, the coloring is as follows: arbitrarily label the points with $\{1,2,\ldots,n\}$; for $i=1,2,\ldots,n-2$, color all segments (i,j), where $i<j$, using color i; use color $n-2$ to color $(n-1,n)$ as well.

We now prove the upper bound for points in convex position. Given a set S of points in convex position, we define the *boundary-distance* between any two points $p,q \in S$ as the minimum number of edges between p and q on the boundary of the convex hull of S; that is, $d(p,q)=1$ if p and q are adjacent, and so on. Label the n points which define $D(n)$ as $\{1,2,\ldots,n\}$. In order to produce the required coloring of $D(n)$ we proceed recursively. For $r<n$, let $D(n,r)$ be the subgraph of $D(n)$ induced by the segments (i,j) such that $d(i,j)>r$. Notice that $D(n,0)=D(n)$ and that $D(n,r)$ is empty if $r \geq n/2$. The key point is the following claim.

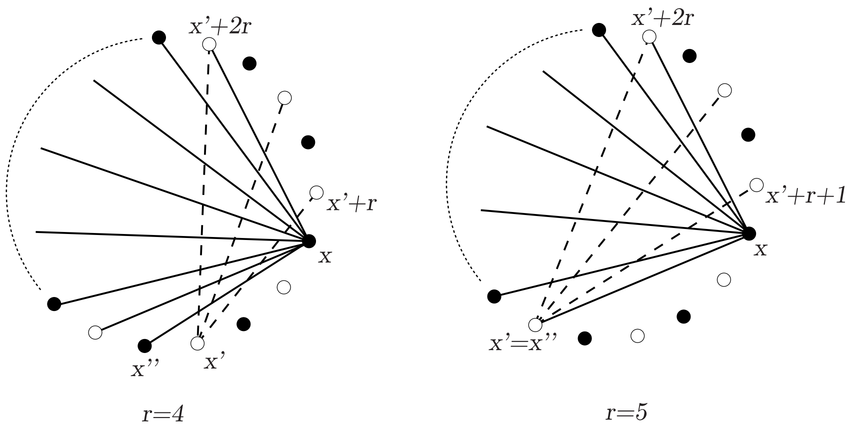

Figure 2: Points in S are white, points in T are black. Sets A_x are drawn with solid lines, sets B_x with dashed lines.

Claim 1 *If $D(n,r)$ can be colored with c colors, then $D(2n,r-1)$ can be colored with $c+n$ colors.*

Proof. Assume we have a c-coloring of $D(n, r)$. Let T be a set of n new points such that $S \cup T$ is in convex position and the points of S and T alternate along the convex hull. That is, we relabel $S = \{1, 3, \ldots, 2n-1\}$ and $T = \{2, 4, \ldots, 2n\}$. Then we can consider $D(n, r)$ as a subgraph of $D(2n, r-1)$, the latter being defined on $S \cup T = \{1, 2, \ldots, 2n\}$. Our task is to color the segments of $D(2n, r-1)$ not in $D(n, r)$ with only n additional colors.

For a new point $x \in T$, let $x' = x - (r-1)$ if r is even, and $x' = x - r$ if r is odd. Additions in this proof are modulo n. Notice that in both cases $x' \in S$. Also, let $x'' = x - r$ and notice that $x'' = x'$ if r is odd (see Fig. 2 for an example). Define also the sets of segments

$$A_x = \begin{cases} \{(x, x'')\} \cup \{(x, y) \mid d(x, y) \geq r + 1\} & \text{for } r \text{ even,} \\ \{(x, x')\} \cup \{(x, y) \mid d(x, y) \geq r\} & \text{for } r \text{ odd.} \end{cases}$$

where the boundary-distance $d(x, y)$ is with respect to $S \cup T$, and

$$B_x = \begin{cases} \{(x', x' + r), (x', x' + r + 2), \ldots, (x', x' + 2r)\} & \text{for } r \text{ even,} \\ \{(x', x' + r + 1), (x', x' + r + 3), \ldots, (x', x' + 2r)\} & \text{for } r \text{ odd.} \end{cases}$$

Finally, set $C_x = A_x \cup B_x$. Notice that segments in B_x always join points in S. It is immediate to check that any two segments of C_x intersect, so that C_x is an independent set in $D(2n, r-1)$ and can be colored with a single color (refer again to Fig. 2).

Then the coloring of $D(2n, r-1)$ is as follows. Start with point $2 \in T$ and color C_2 with a new color. Then color C_4 with a second new color, and continue in this way up to C_{2n}, using a total of n new colors. At some point, we shall be coloring a segment in some C_x already colored in a previous step; it does not matter, the segment gets the last color received in the process. Since this is a correct coloring scheme, it only remains to see that all the segments in $D(2n, r-1)$ have been colored. There are three cases.

1. Segments joining points in S at boundary-distance greater than or equal to $2r + 2$ are already colored by the initial coloring of $D(n, r)$.

2. Segments joining points in T belong to one of the A_x and have been colored; the same applies to segments (i, j) with $i \in S, j \in T$ and $d(i, j) \geq r$.

3. Finally, segments joining points in S at boundary-distance at most $2r$ belong to one of the B_x and thus have been colored.

This conclude s the proof of the claim. □

To prove the upper bound, suppose first that n is power of 2: $n = 2^k$, where $k \geq 2$. Let x be the smallest positive integer for which $x \geq 2^{k-x}/2 = 2^{k-x-1}$.

Claim 2 $x \leq k - \lceil \log k \rceil + 1$.

Proof. It is enough to show that for $x = k - \lceil \log k \rceil + 1$, we have

$$x \geq 2^{k-x-1} \tag{1}$$

(since LHS is increasing in x, and RHS is decreasing in x). Let $k = 2^p + r$, where $1 \leq r \leq 2^p$, and $p \geq 1$. Then (1) is equivalent to

$$k - \lceil \log k \rceil + 1 \geq 2^{\lceil \log k \rceil - 2},$$

or

$$2^p + r - p - 1 + 1 \geq 2^{p+1-2}.$$

This amounts to verifying that $2^{p-1} \geq p - r$, which follows from $2^{p-1} \geq p - 1 \geq p - r$. $\square$

Notice that by the choice of x, $D(2^{k-x}, x)$ is empty. Applying Claim 1 repeatedly, we arrive at a coloring of $D(2^k, 0) = D(n)$ using a number of colors not more than

$$2^{k-1} + 2^{k-2} + \cdots + 2^{k-x+1} + 2^{k-x} = 2^k - 2^{k-x}.$$

By Claim 2,

$$2^{k-x} \geq 2^{k-k+\lceil \log k \rceil - 1} = 2^{\lceil \log k \rceil - 1} \geq 2^{\log k - 1} = \frac{k}{2}.$$

Hence the total number of colors used is at most $2^k - k/2$. Thus for $n = 2^k$,

$$d_c(n) \leq 2^k - \frac{k}{2} = n - \frac{\log n}{2}.$$

For general n, let $k = \lfloor \log n \rfloor$ and $m = n - 2^k$. We can color $D(2^k)$ with at most $2^k - k/2$ colors and use m additional colors for the segments with endpoints in the m additional points. The total number of colors used is not more than

$$2^k - \frac{k}{2} + m = n - \frac{k}{2} = n - \frac{\lfloor \log n \rfloor}{2}.$$

Finally we treat the case where S is a set of n points in general position. By the well-known Erdős-Szekeres theorem, S contains a subset S' of points in convex position, where $m = |S'| \geq \log n/2$. By the previous proof, we can color the induced subgraph $D(S')$ of $D(S)$ with $m - \lfloor \log m \rfloor/2$ colors. For every point x in $S \backslash S'$ we choose a new color c_x and all edges incident with x are colored with c_x. This gives a proper coloring of $D(S)$. The total number of colors used is at most

$$\left(m - \frac{\lfloor \log m \rfloor}{2}\right) + (n - m) = n - \frac{\lfloor \log m \rfloor}{2} \leq n + \frac{1}{2} - \frac{\lfloor \log \log n \rfloor}{2},$$

and the upper bound in *(ii)* follows.

3 Proof of Theorem 2

Two segments can intersect either at an interior point or at a common endpoint. Thus for points in convex position, the clique number of $I(S)$ satisfies $\omega(I(S)) \geq n$. Indeed, if p is an arbitrary point of S, denote by a and b its two adjacent vertices in say, clockwise order; then the set of $n - 1$ segments adjacent to p, together with the segment (a, b) forms a set of n pairwise intersecting segments. This proves that $i_c(n) \geq n$ and $i(n) \geq n$.

Next we show the upper bounds. First we analyze the case of points in convex position and verify that $i_c(n) \leq n$. We make use of the following well-known fact.

Lemma 1 *The edge set of a complete geometric graph whose vertices form a regular n-gon can be partitioned into n matchings, each consisting of parallel segments.*

Since the crossing pattern of the edge set of a complete geometric graph whose vertices are in convex position is the same as the one of a regular polygon, the upper bound follows.

For points in general position, we prove that $i(n) = O(n^{3/2})$, by using another result on geometric graphs.

Theorem 5 (G. Tóth, [12]) *For any $k < n/2$, a geometric graphs on n vertices with no $k+1$ pairwise disjoint edges has at most 2^9k^2n edges.*

Start with the complete geometric graph G on a given set of n points, and repeatedly remove a large (independent) set of pairwise disjoint edges, until the graph becomes empty. Color each such set using a different color. This is a proper coloring and it only remains to show that the number of independent sets can be bounded as claimed.

The process consists of at most $\lceil \log(n^2) \rceil$ steps, numbered with $i = 2, \ldots, \lceil \log(n^2) \rceil + 1$. In step i, the current graph, still denoted by G, has m edges, where

$$\frac{n^2}{2^i} < m \leq \frac{n^2}{2^{i-1}}. \tag{2}$$

Set

$$k = \left\lfloor \sqrt{\frac{n}{2^{i+9}}} \right\rfloor,$$

and apply Theorem 5, to find and remove $k+1$ disjoint edges. This is done repeatedly until m fails to satisfy (2), and the process continues with step $i+1$.
The number of independent sets of edges removed in step i is at most

$$\frac{n^2}{2^i} \left(\sqrt{\frac{n}{2^{i+9}}} \right)^{-1} = n\sqrt{n}\, 2^{9/2} 2^{-i/2}.$$

Hence the total number of colors used is not more than

$$n\sqrt{n}\, 2^{9/2} \sum_{i=2}^{\infty} 2^{-i/2} = O(n\sqrt{n}).$$

Observations. Let $e_k(n)$ be the smallest number such that any geometric graph with n vertices and $m > e_k(n)$ edges contains $k+1$ pairwise disjoint edges, where $k < n/2$ (cf. [13, 12]). Theorem 5 improved on previous upper bounds, where the dependence on k was in the fourth [11], and respectively in the third power [13]. The best lower bounds on $e_k(n)$ are $e_k(n) \geq kn$ [7], and $e_k(n) \geq \frac{3}{2}(k-1)n - 2k^2$ [13], and it is believed that $e_k(n) = O(kn)$. Assuming that $e_k(n) = O(kn)$ holds, and using this bound instead of the bound $e_k(n) = O(k^2n)$ we used in the proof of Theorem 2, would give $i(n) = O(\sum_{i=2}^{i=\log n^2} n) = O(n \log n)$, which is still above our linear lower bound.

In fact, it is reasonable to expect that one can partition the set of edges of a complete geometric graph into a small number of non-crossing matchings (composed of pairwise disjoint edges), if the process is carried out carefully, and does not make use of the bound in Theorem 2, which holds for *any* geometric graph with sufficiently many edges. However, this goal remained elusive to us.

A simpler recursive decomposition scheme which gives a somewhat weaker upper bound $i(n) = O(n^{\log 3}) \approx O(n^{1.59})$ is as follows. Let $b(n)$ denote the minimum number of non-crossing matchings into which the edge set of a complete geometric bipartite graph can be partitioned, with parts containing $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ vertices respectively, and in which the two parts are separated by a line. Let $c(n)$ denote the minimum number of non-crossing matchings into which the edge set of a complete geometric graph on n vertices can be partitioned. Since $c(n)$ satisfies the recurrence

$$c(n) \leq c(\lfloor n/2 \rfloor) + c(\lceil n/2 \rceil) + b(n),$$

it is enough to prove such an upper bound on $b(n)$. We proceed as follows. Let l be a vertical line which yields a balanced partition of S, and h another line which simultaneously cuts the two sets, on the left and right sides of l, into two halves of equal size (see Fig. 3). Then $b(n)$ satisfies the recurrence

$$b(n) \leq 3b\left(\left\lceil \frac{n}{2} \right\rceil\right),$$

which yields the claimed bound $b(n) = O(n^{\log 3})$.

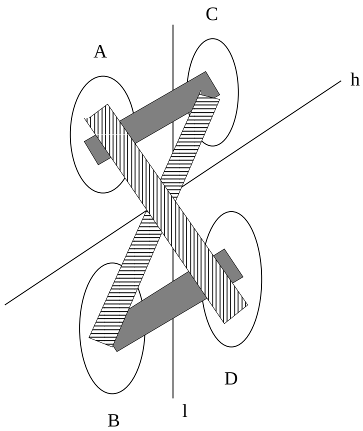

Figure 3: A recursive decomposition scheme for the edges a complete bipartite geometric graph.

4 Proof of Theorem 3

The upper bound $w_c(n) = O(n \log n)$ follows from the following explicit coloring. Assign a different color to each edge of length 1 (the length of a segment is the boundary-distance between its two endpoints, as in Section 2). Next group the edges of length 2 in $n/2$ crossing pairs and assign a new color to each pair. Proceed in this way until all edges of length $n/2$ get the same color (for simplicity we are ignoring integer parts). The total number of colors used is $n + (n/2) + (n/3) + \cdots = \Theta(n \log n)$.

We now prove that $w_c(n) = \Omega(n \log n)$. Let us consider any proper coloring $\mathcal{C}$ of $W(S)$, let c be the number of colors in $\mathcal{C}$. Let us take as colors the integer numbers $1, \ldots, c$. For each color $i = 1, \ldots, c$, let S_i be the set of segments in $\mathcal{C}$ with color i, and let L_i be the finite ordered sequence of their lengths. Notice that every length in L_i is at least $|L_i|$, because a segment of length t can participate in families of at most t pairwise crossing segments. Let $F_1 = (L_1, \ldots, L_c)$ be the c-tuple having as elements these sequences.

Observe that a similar construction for the explicit coloring scheme described in the first part of this proof in order to establish the upper bound would give the $|F|$-tuple of sequences

$$F = (\{1\}, \{1\}, \ldots, \{1\}, \{2, 2\}, \{2, 2\}, \ldots \{2, 2\}, \{3, 3, 3\}, \ldots)$$

(assuming that n is even, as otherwise we would have a $\{2\}$ alone, and similar integer part considerations would apply to the other numbers).

From F_1 we define a new c'-tuple F_2, with $c' \leq c$, having as elements sequences of integer numbers as follows. Let F_1' be the subset of elements of F_1 which are sequences that contain the integer number 2; some of them, say a total of v_1, will contain one 2 and possibly some other integer different from 2, and some of them, v_2, will contain exactly two 2's (therefore $|F_1'| = v_1 + v_2$). Let s be the number of integers different from 2 involved in sequences belonging to F_1', and notice that $s \leq v_1$. We replace these

$v_1 + v_2$ sequences in F_1 by forming $\lceil (v_1 + 2v_2)/2 \rceil$ pairs with the 2's (leaving possibly one 2 alone) and forming $\lceil s/2 \rceil$ pairs with the elements different from 2 (leaving possibly one of them alone); i. e., we replace $\{2\}, \{2\}$ by $\{2, 2\}$ and $\{2, a\}, \{2, b\}$ by $\{2, 2\}, \{a, b\}$. As we have

$$\left\lceil \frac{v_1 + 2v_2}{2} \right\rceil + \left\lceil \frac{s}{2} \right\rceil \leq \left\lceil \frac{v_1 + 2v_2}{2} \right\rceil + \left\lceil \frac{v_1}{2} \right\rceil \leq v_1 + v_2 + 1,$$

we see that $|F_2| \leq |F_1| + 1$.

Notice that we do not claim that the c'-tuple F_2 corresponds to any coloring of $W(S)$, it is just a c'-tuple of sequences of numbers in the abstract sense. Nevertheless the crucial property that each number appears in a sequence the right number of times is maintained (i.e., t does not belong to any sequence of length greater than t).

From F_2 we again define a c''-tuple F_3, with $c'' \leq c'$: Let F_2' be the subset of elements of F_2 which are sequences that contain the integer number 3; some of them, say a total of w_1, will contain one 3 and possibly some other integers different from 3, some of them, w_2, will contain two 3's and possibly some other integer, and some of them, w_3, will contain exactly three 3's. Let s be the number of integers different from 3 involved in sequences belonging to F_2'; by the definition we have $s \leq 2w_1 + w_2$. We replace these $w_1 + w_2 + w_3$ sequences in F_2 by forming $\lceil (w_1 + 2w_2 + 3w_3)/3 \rceil$ triplets with the 3's (leaving possibly one group with one or two of them) and forming $\lceil s/3 \rceil$ triplets with the elements different from 3 (again leaving possibly one group with one or two elements). As we have

$$\left\lceil \frac{w_1 + 2w_2 + 3w_3}{3} \right\rceil + \left\lceil \frac{s}{3} \right\rceil \leq \left\lceil \frac{w_1 + 2w_2 + 3w_3}{3} \right\rceil + \left\lceil \frac{2w_1 + w_2}{3} \right\rceil \leq w_1 + w_2 + w_3 + 1,$$

we see that $|F_3| \leq |F_2| + 1 \leq |F_1| + 2$.

Iterating this process for the lengths $4, \ldots, \lfloor n/2 \rfloor$we would end up with the $|F|$-tuple of sequences F. Therefore $c + \lfloor n/2 \rfloor \in \Omega(n \log n)$, for any proper coloring $\mathcal{C}$ of $W(S)$, where $|\mathcal{C}| = c$ and hence $w_c(n) \in \Omega(n \log n)$ as claimed.

A subquadratic upper bound on $w(n)$ is implied by a result analogous to Theorem 5, and the proof is similar to that of the upper bound on $i(n)$ in Theorem 2.

Theorem 6 [10] *For any $k < n/2$, a geometric graphs on n vertices with no $k+1$ pairwise crossing edges has at most $3n(10 \log n)^{2k-2}$ edges.*

Start with the complete geometric graph G on a given set of n points, and repeatedly remove a large (independent) set of pairwise crossing edges, until the graph has roughly $n^{7/4}$ edges (this threshold is quite arbitrary). Color each such set using a different color. Then color each of the remaining edges with a new color. We obtain in this way a proper coloring.

The process consists of at most $\lfloor (\log n)/4 \rfloor - 1$ steps, numbered with $i = 2, \ldots, \lfloor (\log n)/4 \rfloor$. In step i, the current graph, still denoted by G, has m edges, where

$$\frac{n^2}{2^i} < m \leq \frac{n^2}{2^{i-1}}. \tag{3}$$

So after step $i = \lfloor (\log n)/4 \rfloor$, the number of edges of G is at most

$$\frac{n^2}{2^{\lfloor (\log n)/4 \rfloor}} \leq 2n^{7/4}.$$

Set

$$k = \left\lfloor \frac{\log n}{8 \log \log n} \right\rfloor,$$

and apply Theorem 6, to find and remove $k+1$ pairwise crossing edges. This is done repeatedly until m fails to satisfy (3), and the process continues with step $i+1$.

Without loss of generality we can assume that $n \geq 2^{16}$. A routine calculation shows that for $i = 2, \ldots, \lfloor (\log n)/4 \rfloor$, we have

$$\frac{n^2}{2^i} \geq 3n(10 \log n)^{2k-2}.$$

The number of independent sets of edges removed in step i is at most

$$\frac{n^2}{2^i} \left(\frac{\log n}{8 \log \log n} \right)^{-1},$$

so the total total number of colors used is at most

$$2n^{7/4} + 8 \left(\sum_{i=2}^{\infty} 2^{-i} \right) n^2 \cdot \frac{\log \log n}{\log n} = O\left(n^2 \cdot \frac{\log \log n}{\log n} \right),$$

as required. This completes the proof of Theorem 3.

Acknowledgement. We thank János Pach for pertinent remarks and suggestions, and an anonymous reviewer for simplifications made to one of our proofs.

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