Point set stratification and Delaunay depth.

Manuel Abellanas [a,1], Mercè Claverol [b,*,2] Ferran Hurtado [c,2]

[a] *Departamento de Matemática Aplicada, Universidad Politécnica de Madrid, Facultad de Informática, Boadilla del Monte, 28660 Madrid, Spain*

[b] *Departament de Matemàtica Aplicada IV, Universitat Politècnica de Catalunya, EPSEVG. Avda. Víctor Balaguer s/n 08800 Vilanova i La Geltrú, Spain*

[c] *Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Jordi Girona 1-3, 08034 Barcelona, Spain*

Abstract

In the study of depth functions it is important to decide whether a depth function is required to be sensitive to multimodality. An analysis of the Delaunay depth function shows that it is sensitive to multimodality. This notion of depth can be compared to other depth functions such as the convex and location depths. The stratification that Delaunay depth induces in the point set (layers) and in the whole plane (levels) is investigated. An algorithm for computing the *Delaunay depth contours* associated with a point set in the plane is developed. The worst case and expected complexities of the algorithm are $O(n \log^2 n)$ and $O(n \log n)$, respectively. The depth of a query point p with respect to a data set S in the plane is the depth of p in $S \cup \{p\}$. The Delaunay depth can be computed in $O(n \log n)$ time, which is proved to be optimal, when S and p are given in the input.

Key words: Tukey depth, halfspace depth, convex depth, Delaunay depth, depth contours, layers

---

[*] Corresponding author

Email addresses: mabellanas@fi.upm.es (Manuel Abellanas), merce@ma4.upc.edu (Mercè Claverol), Ferran.Hurtado@upc.edu (Ferran Hurtado).

[1] Partially supported by MCYT-TIC2003-08933-C02-01, and CAM S0505/DPI/023.

[2] Partially supported by DURSI 2005SGR00692, MCYT BFM2003-0368 and MEC MTM2006-01267.

1 Introduction

In multivariate analysis, classical parametric methodologies are sensitive to outlying data points and rely on assumptions about the underlying distribution (as normality or some kind of symmetry). Data depth has been considered as a measure of how deep or central a given point is with respect to a multivariate distribution. Recently nonparametric methods have been developed based on the concept of data depth [21]. The affine invariance property of data depth and the spatial ordering of the sample points leads to the introduction of different methods for analyzing multivariate distributional characteristics. A survey of statistical applications of multivariate data depth is provided by Liu et al. [21]. Several notions of depth have been considered: *location depth*, also known as *halfspace depth* or *Tukey depth* [39], *convex depth* or *convex hull peeling depth* [15], [5], *Delaunay depth* [14], *Oja depth* [26], *simplicial depth* [20], and *regression depth* [32]. A classification of multivariate data depths based on their statistical properties is proposed by Zuo [42].

Every notion of depth of a point with respect to a point set S gives rise to a partition of the set S into *layers* and also to a partition of the whole plane into *levels*. The layers are graphs having as vertex set the points of S having the same depth. Edges are defined differently depending on the depth used. The levels are the regions of points in the plane with the same depth with respect to S. The boundaries of the levels are known as *depth contours* and provide a quick and informative overview of the shape and some properties of the point set. For this reason, Tukey suggested the use of depth contours as an appropriate tool for data visualization [39].

Obviously, for any specific purpose of a given statistical analysis, certain notions of depth may be more suitable than others. Okabe et al. [27](p. 363) suggest the value of comparing *Delaunay depth* to other depths. In this paper we focus on Delaunay depth and compare the properties of layers and levels associated to finite sets of points in the plane to the cases of *convex depth* and *location depth*. A thorough study of these three measures is presented in [8].

The Delaunay triangulation $DT(S)$ is one of the possible *proximity graphs* describing vicinity in a given point set S [18]. The Delaunay depth with respect to S of a point $p \in S$ is the length of the shortest path from p to the convex hull of S in $DT(S)$, augmented by 1. This is a matter of taste, as one may want to have the extreme points having depth 1 or, alternatively, depth 0. A systematic study of depths, particularly in proximity graphs, is the topic of the series of papers [16], [17], [28], [29], [30] and [31], where the proximity graph depth of a point $p \in S$ is defined to be the number of edges in the proximity graph of S that must be traversed to reach the convex hull of S, and the proximity graph depth of a point $x \notin S$ with respect to S is defined to be the length of the shortest path from x to the convex hull of $S \cup \{x\}$ in the proximity graph of $S \cup \{x\}$.

A main concern in current theoretical research on data depth is to find the *depth contours* and *central regions* by which the underlying distribution may be characterized. In the discrete geometry literature, the *centerpoint* is any point with location depth greater than or equal to $\lceil n/(d+1) \rceil$ in $\mathbb{R}^d$ (where n is the number of points in the input). The *median* is defined as a point with maximum depth. The set of points of maximum depth is a connected set in the case of location depth or convex depth. The situation is different for Delaunay depth, as shown later; on the other hand in this case it might be interesting to consider all local maxima, aiming to the multimodality features of the underlying set of points. Delaunay depth works well on general distributions and is better than other depths in some respects, since it is sensitive to the existence of clusters and neighborhood relations between the points. Many interpolation methods are based on Voronoi diagrams and Delaunay triangulations as a natural neighbor interpolation method [36]. A selection of clustering methods is presented in [38]. Different schemes have been proposed for cluster representation; for example, in [13] a hierarchical clustering algorithm is developed, and in [25] another clustering algorithm based on closest pairs is described.

When the median is not a unique point, it is often taken to be the centroid of the deepest region. In particular, and regarding the statistical applications, several medians have been explicitly considered: the location median, the convex depth median, the maximum simplicial depth median, and the minimum Oja depth median, as well as a line or a flat with maximum regression depth. An overview of several multivariate medians and their basic properties can be found in [37]. The location median can be used as a point estimator for the data set, and it is also robust against outliers, does not rely on distances, and is invariant under affine transformations. The location depth and the corresponding median have good statistical properties as well [6]. Rousseeuw and Struyf present a complete survey about depth, medians, and related measures in [34]. There is also a survey on data depth by Aloupis [1].

After introducing the basic definitions in Section 2, we give an algorithm in Section 3 for computing the *Delaunay depth contours* (boundaries of the levels), associated with a point set in the plane. Therefore, we will know the *Delaunay median* after computing all the levels within the running time of the algorithm, which is $O(n \log^2 n)$ worst case or $O(n \log n)$ expected. We also study and compare the complexity of the layers and levels of the convex, location and Delaunay depths. In particular, we see that the depth of a point p with respect to a set of data $S = \{s_1, \ldots, s_n\}$ can be found in $O(n \log n)$ time. Lower bounds for this kind of problem have attracted significant attention, and in Section 4 we carry out a study similar to those by Aloupis et al. in [2] and [3], proving an $\Omega(n \log n)$ lower bound for Delaunay depth computation.

2 Preliminaries

Let S be a set of n points in the plane; their convex hull will be denoted by $CH(S)$ and the boundary of this set by $\partial CH(S)$. The *Delaunay graph* of S, denoted by $DG(S)$, is the graph with vertex set S, two points p and q being adjacent when there is some circle through p and q whose interior is empty of points from S. In general this graph may have quadratic size because a subset of S consisting of cocircular points such that the circle through them contains no other point in S defines a complete subgraph of $DG(S)$. When this does not happen, the graph $DG(S)$ is a triangulation of S, called the Delaunay triangulation and denoted by $DT(S)$.

Throughout this paper we assume that all point sets S that are given as input data are in *non-degenerate position*, in the sense that not all points in S are collinear and if any four of them are cocircular, then the interior of the circle through them is not empty of points from S. In other words, we assume that point sets S have a properly and uniquely defined Delaunay triangulation. We discuss the options for the situations in which this condition is not fulfilled, in Section 5.

The fact that S is in non-degenerate position is crucial for the complexity of the combinatorial structures and the running times of the algorithms. In particular, for $p \notin S$, the set $S \cup \{p\}$ will be in degenerate position if and only if p lies on one of the circles that circumscribe the triangles in $DT(S)$, and even in this case $DG(S \cup \{p\})$ will have linear size, because for each of those circles we may get at most one extra edge.

Any generic depth of $p \in S$ with respect to S is denoted by $d_S(p)$ and the levels and layers of S by $Lev_i(S)$ and $Lay_i(S)$, respectively. For the specific cases that we study, we add superscripts as indicated in the following paragraphs.

Let S' be the subset of S consisting of the elements that lie on the boundary of $CH(S)$. The *convex depth* of a point $p \in S$, is defined recursively as follows: if $p \in S'$ then its convex depth, denoted by $d_S^C(p)$, is equal to 1. Otherwise $d_S^C(p) = d_{S \setminus S'}^C(p) + 1$. The i-th *convex layer* of S, $Lay_i^C(S)$, is the geometric graph with vertex set $S_i = \{x \in S | \, d_S^C(x) = i\}$ and the adjacencies given by the portions of the boundary of $CH(S_i)$ between consecutive vertices; see Figure 1. The layers decompose the plane in nested regions; the points p that lie between $Lay_{i-1}^C(S)$ and $Lay_i^C(S)$ (including the boundary of $CH(S_i)$) are said to have *convex depth* i with respect to S; this is denoted by $d_S^C(p) = i$. The points on the boundary of $CH(S)$ or in its exterior get depth 1. The set

	$d_S(p)$ (Depth)	$Lay_i(S)$ (Layer i)	$Lev_i(S)$ (Level i)
Convex	If p is not interior to $CH(S)$, then $d_S(p) = 1$ else $d_S(p) = d_{S\setminus(S\cap\partial CH(S))}(p) + 1$	$Lay_i(S) = CH(S_i)$ considered as an embedded cycle graph with vertex set S_i and edges along the boundary of $CH(S_i)$, where $S_i = \{x \in S \mid d_S(x) = i\}$	$Lev_i(S)$= $\{x \in \mathbb{R}^2 \mid d_S(x) = i\}$
Location	$d_S(p) = j, \quad j \leq \lfloor \lvert S\rvert/2 \rfloor \Leftrightarrow$ $\exists H \mid p \in \partial H$, $\lvert H \cap S\rvert = j$ $\nexists H' \mid p \in \partial H'$, $\lvert H' \cap S\rvert < j$ H, H' closed half-planes		
Delaunay	$d_S(p) =$ distance from p to $CH(S \cup \{p\})$ +1, in $DG(S \cup \{p\})$	$Lay_i(S)$ =subgraph of $DT(S)$ induced by S_i $S_i = \{x \in S \mid d_S(x) = i\}$	

Table 1: Basic definitions (singular cases are determined in the text)

of points on the plane that have depth i with respect to S is called the i-th *convex level* and is denoted by $Lev_i^C(S)$.

For values of $j \leq \lfloor |S|/2 \rfloor$, we say that the *location depth* of a point p on the plane is $d_S^L(p) = j$ if and only if there is a closed half-plane containing p on its boundary that contains exactly j points of S, and no other closed half-plane passing through p contains less than j points of S. The i-th *location layer* of S, $Lay_i^L(S)$, and the i-th *location level* of S, $Lev_i^L(S)$, are defined as for convex depth in the preceding paragraph, using d_S^L instead of d_S^C ; see Figure 2. The concept of k-hull introduced by Cole, Sharir and Yap [9] corresponds to $\bigcup_{j \geq k} Lev_j^L(S)$, also know as k-th depth region D_k.

The *Delaunay depth* of $p \in S$, $d_S^D(p)$, is defined to be $d + 1$ when the graph theoretical distance from p to $CH(S)$ in the Delaunay triangulation $DT(S)$ of S is d. The i-th *Delaunay layer* of S, $Lay_i^D(S)$ is the subgraph of $DT(S)$ induced by S_i; see Figure 3.

For any point p on the plane, the *Delaunay depth* of p with respect to S, $d_S^D(p)$, is defined to be $d{+}1$ when the graph theoretical distance from p to $CH(S{\cup}\{p\})$

in the Delaunay graph $DG(S \cup \{p\})$ is d. Notice that in the case $p \in S$, this definition agrees with the definition given in the preceding paragraph, because in this case $S \cup \{p\} = S$ and $DG(S \cup \{p\}) = DG(S) = DT(S)$. We prefer a 2-step definition to keep the original intuition that comes from Delaunay triangulations. The i-th *Delaunay level* for the set S is defined as $Lev_i^D(S) = \{x \in \mathbb{R}^2 | \, d_S^D(p) = i\}$.

In all three cases the *depth of* S is its deepest point. Table 1 summarizes all these definitions.

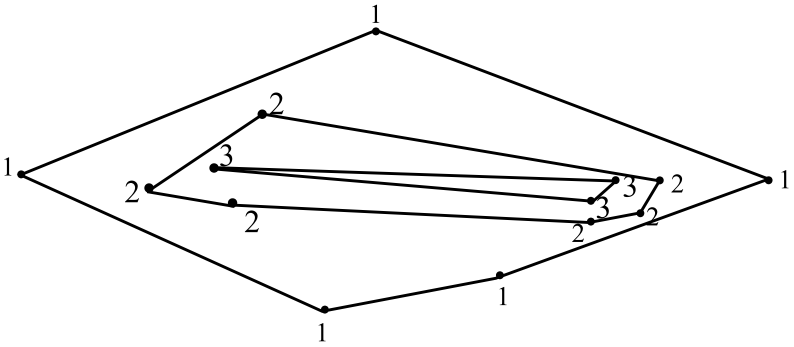

Fig. 1. Convex layers.

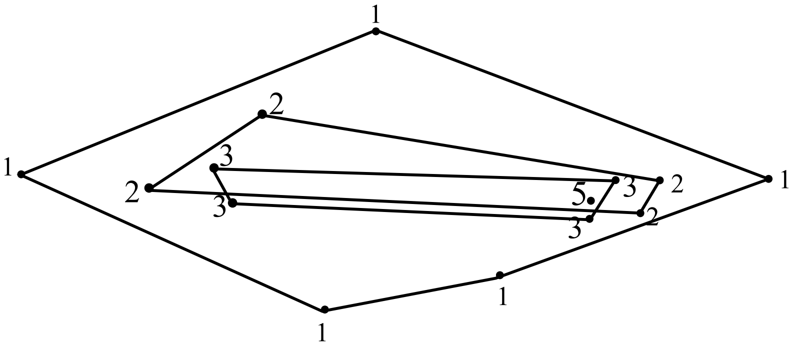

Fig. 2. Location layers.

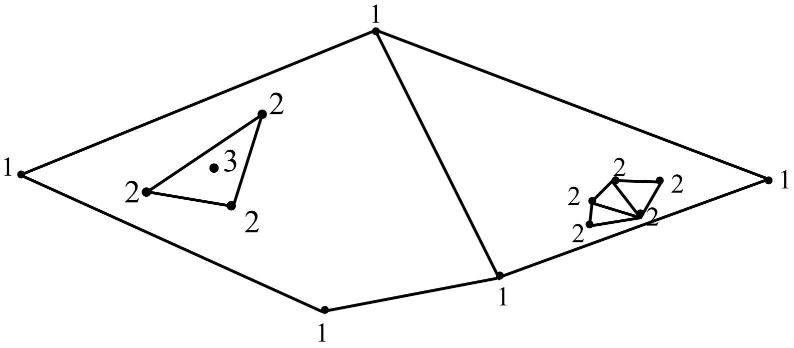

Fig. 3. Delaunay layers.

3 Point set stratification

Given a set S of n points in the plane, the convex layers can be constructed with Chazelle's optimal $\Theta(n \log n)$ algorithm [7]. Convex layers form a sequence of nested convex polygons defining a partition of the plane into regions, which coincide with the levels; see Figures 1 and 4. Therefore layers and levels have linear complexity in the convex depth case and can be constructed in optimal $O(n \log n)$ time.

The line-sweep for computing all location depth contours was initially described by Matousek [22]. For location depth, a worst case optimal algorithm for computing all $Lev_i^L(S)$ in $O(n^2)$ time (where $n/3 \leq i \leq n/2$) is obtained by using topological sweep in the dual arrangement of lines (see [8], [23]). The boundaries of the levels, in this case, form a sequence of nested convex polygons. The vertices of $Lay_i^L(S)$ are in convex position and belong to the boundary of $Lev_i^L(S)$, but this boundary can also have other vertices not in S; see Figure 5. Some layers can be empty and different layers can cross each other; see Figure 2. While the size of levels may reach $O(n^2)$, the size of the layers is $O(n)$. The layers in the location depth case can be computed using the mentioned $O(n^2)$ sweep algorithm. However to our knowledge, it is an open problem to construct them in less time or to prove a quadratic lower bound for the problem.

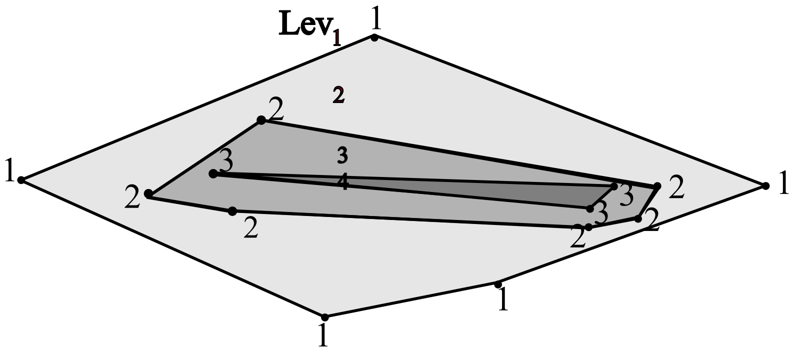

Fig. 4. Convex levels.

In the remainder of this section, we systematically explore Delaunay depth, which is the least studied of all depths.

In the Delaunay depth case, all the layers $Lay_i^D(S)$, where $i \leq n/3$, can easily be found by visiting $DT(S)$ in linear time once constructed. This requires $O(n \log n)$ time (Figure 3). Notice that one layer can have more than one connected component. First, we show some properties of Delaunay layers that allow us to obtain the levels easily and also prove other results, such as that the overall size of the levels is $O(n)$. Next, we study the number of connected

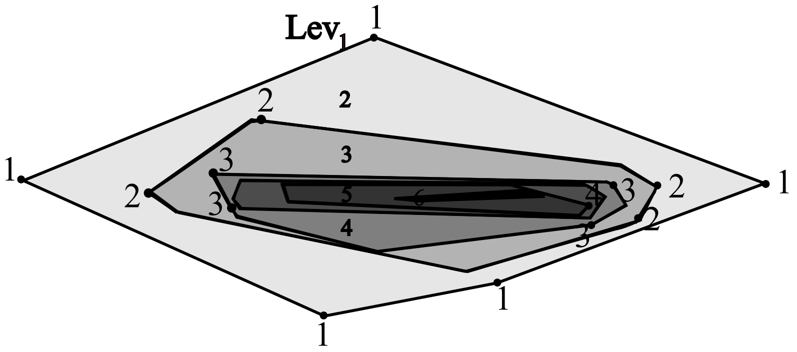

Fig. 5. Location levels.

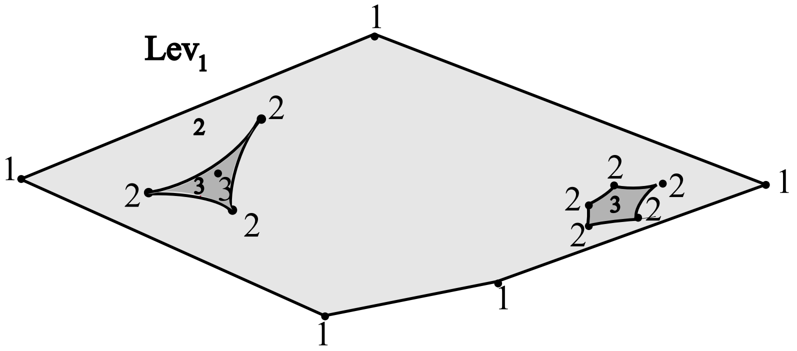

Fig. 6. Delaunay levels.

components that $\bigcup Lay_i^D(S)$ can have.

Proposition 1 *Let S be a set of Delaunay depth greater than 1. The points of S that are in the interior of any cycle C_i of $Lay_i^D(S)$ have depth greater than i.*

Proof. Let $p \in S$ be a point in the interior of a cycle C_i of $Lay_i^D(S)$. From the definition of Delaunay depth, we know that p must have some adjacency of depth $d_S^D(p) - 1$. The points adjacent to p are vertices of C_i or they are in the interior of C_i.

If we suppose the assertion of the proposition is false, then $d_S^D(p) \leq i$. Thus there exists a point q adjacent to p, with $d_S^D(q) = d_S^D(p) - 1$ and interior of C_i. Recursively it follows that there is at least a point of depth 1 in the interior of C_i, which is impossible. We conclude that all points of S that are in the interior of C_i have depth greater than i. $\square$

Lemma 2 *Let S be a set of Delaunay depth greater than 1. Any cycle of $Lay_i^D(S)$ without chords has at most one connected component of $Lay_{i+1}^D(S)$ in its interior.*

Proof. Let C_i be a cycle of $Lay_i^D(S)$ without chords.

Suppose, contrary to our claim, that there is more than one connected component of $Lay^D_{i+1}(S)$ in the interior of C_i. By the above assumption, we first prove that there is a vertex $v_i \in C_i$ that is adjacent to some vertices belonging to different connected components of $Lay^D_{i+1}(S)$ in the interior of C_i; see Figure 7. Let $v_i^1, v_i^2, \ldots, v_i^n$ be the vertices of C_i in clockwise order along the cycle. We study the adjacencies of these vertices in the interior of C_i. Note that these adjacencies have depth equal to $i+1$ (by applying Proposition 1, and the fact that their depth cannot differ from i by more than 1). Furthermore, all the vertices of $Lay^D_{i+1}(S)$ in the interior of C_i must have at least one adjacency in C_i.

We move along C_i following the adjacencies: while the adjacencies are of the same connected component we advance one vertex in C_i. We want to find different connected components in the adjacencies. There are two possibilities:

(1) There is a vertex v_i^j that is adjacent to some vertices of different connected components of $Lay^D_{i+1}(S)$ in the interior of C_i.
(2) There are vertices v_i^j and v_i^{j+1}, for some j, whose adjacencies are in different components of $Lay^D_{i+1}(S)$; see Figure 8.

But in the second case, we see that either v_i^j or v_i^{j+1} also have adjacencies in different components of $Lay_i(S)$ (this is similar to the first case). In order to prove this, consider the point v that forms a triangle in $DT(S)$ with v_i^j and v_i^{j+1} and is not exterior to C_i. This point v is of depth $i+1$; it cannot be i because then $v_i^j v$ and $v_i^{j+1} v$ would be chords of C_i, which contradicts the hypothesis of the proposition. Hence, $v \in Lay^D_{i+1}(S)$ in the interior of C_i, but v cannot belong at the same time to the different connected components where v_i^j and v_i^{j+1} have adjacencies.

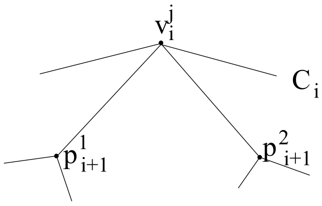

Fig. 7. There is a vertex $v_i^j \in C_i$ adjacent to some vertices belonging to different connected components.

We have proved that there is a vertex $v_i^j \in C_i$ that is adjacent to some vertices of different components of $Lay^D_{i+1}(S)$, denoted by p^1_{i+1}, p^2_{i+1} as in Figure 7. Consider the path in $DT(S)$ between p^1_{i+1} and p^2_{i+1} formed by a sequence of vertices, all adjacent to v_i^j. Note that this sequence can only be formed by points of depth $i+1$: there is no point with depth $i+2$ because this point would be adjacent to v_i^j which has depth i, and also there is no point of depth equal to i because this point would form a chord with v_i^j, which contradicts

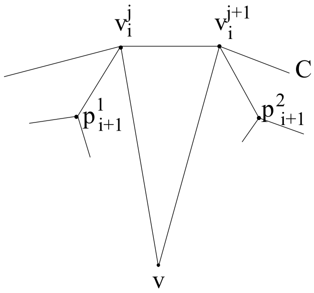

Fig. 8. The vertices v_i^j and v_i^{j+1} of C_i have adjacencies of different components of $Lay^D_{i+1}(S)$.

the assumptions.

The layer $Lay^D_{i+1}(S)$ is formed by the subgraph induced in $DT(S)$ by the points with the same depth, so all the points adjacent to v_i^j, between p^1_{i+1} and p^2_{i+1}, are in the same connected component, a contradiction.

Hence we conclude that any cycle of $Lay^D_i(S)$ without chords does not contain more than one connected component of $Lay^D_{i+1}(S)$ in its interior. $\square$

Lemma 3 *Let S be a set of points in the plane. Let $p \in S$ such that $p \in Lay^D_{i+2}(S)$ and let C_i be a cycle without chords of $Lay^D_i(S)$ that contains p in its interior. Then there is a cycle of $Lay^D_{i+1}(S)$ containing p in its interior.*

Proof. Let T_p be the set of Delaunay triangles that are incident to p. Let C_1 be the cycle that forms the boundary of the union of all triangles in T_p. If all vertices in C_1 have Delaunay depth equal to $i+1$ we are done. Otherwise there is a vertex q of C_1 with $d^D_S(q) \neq i+1$. Add to T_p all Delaunay triangles incident to q and let C_2 be the boundary of the union of triangles in T_p. If all vertices of C_2 have depth $i+1$, C_2 is the desired cycle. The process can be iterated while the cycle contains vertices with depth different to $i+1$. It necessarily ends after a finite number of steps because we are adding Delaunay triangles. Note that vertices in C_j have depth at least $i+1$ because p is contained in C_i (Proposition 1). $\square$

Lemma 4 *Let S be a set of points and let $q \notin S$ be a point in the plane and such that $S \cup \{p\}$ is in non-degenerate position. Let j be the Delaunay depth of q with respect to $S \cup \{q\}$. If the depth of $p \in S$ with respect to S is i with $i < j$, then $d^D_{S\cup\{q\}}(p) = d^D_S(p) = i$.*

Proof. There is a path γ in $DT(S)$ of length $i-1$ connecting p with the convex hull of S. For each edge of this path there exists a circle passing through its two endpoints whose interior does not contain any point from S. None of these circles can cover q, because this would imply $d^D_{S\cup\{q\}}(q) \leq i < j = d^D_{S\cup\{q\}}(q)$. Therefore γ is also a path in $DT(S \cup \{q\})$ and hence $d^D_{S\cup\{q\}}(p) \leq i$.

If there is a path σ in $DT(S \cup \{q\})$ connecting p to the convex hull of $S \cup \{q\}$ with less than $i-1$ edges, it cannot be a path in $DT(S)$, because we would apply to σ the preceding argument given for γ and get $d^D_S(p) < i$; therefore σ must pass through q, which is again a contradiction because then $d^D_{S\cup\{q\}}(q) < j$. $\square$

Proposition 5 *Let S be a set of points in the plane and let p be a point possibly not in S such that $S \cup \{p\}$ is in non-degenerate position. If the Delaunay depth of p with respect to S is $j+1$ $(0 < j \leq f-1$, where f is the depth of $S)$ then there is a cycle of $Lay^D_j(S)$ containing p in its interior.*

Proof. We prove the claim by induction on j with induction hypothesis: if the Delaunay depth of p with respect to S is $k+1$ $(0 < k \leq j-1)$ then there is a cycle of $Lay^D_k(S)$ containing p in its interior.

If $j = 1$, then $Lay^D_1(S \cup \{p\})$ contains p and $Lay^D_1(S \cup \{p\})$ is a cycle with vertices in $S \cap \partial CH(S)$.

First assume that $p \in S$. Let p be a point with depth $j+1$ with respect to S. By the definition of Delaunay depth, there is a vertex $q \in Lay^D_j(S)$ that is adjacent to p. Applying the induction hypothesis to q, there is a cycle of $Lay^D_{j-1}(S)$ containing q and its adjacent vertices (p among them), in its interior. If this cycle of $Lay^D_{j-1}(S)$ has no chords, we can apply Lemma 3 to this cycle and p and we are done. If this cycle has chords, it contains a sub-cycle enclosing p with no chords and we can apply Lemma 3 to p and the sub-cycle.

Now consider the case in which $p \notin S$. The depth of p with respect to S is equal to $j+1$ or equivalently, by definition, the depth of p is $j+1$ in $S \cup \{p\}$. We repeat the above induction proof to p in $S \cup \{p\}$. We conclude that there is a cycle of $Lay^D_j(S \cup \{p\})$ that contains p in its interior. If some point has modified its depth after the addition of p, the modified depth never can be less than the depth of p (which is $j+1$). From Lemma 4 the set of points with depth less than $j+1$ in S is the same set in $S \cup \{p\}$. Hence the cycle of $Lay^D_j(S \cup \{p\})$ that contains p in its interior is also a cycle of $Lay^D_j(S)$. $\square$

As a consequence of Proposition 5, the number of levels for Delaunay depth is equal to the number of layers or to the number of layers plus 1.

Proposition 6 *Let S be a set of n points. The maximum number of connected components of $\bigcup Lay^D_i(S)$ is $\lfloor (n-m+2)/2 \rfloor$, where $m \geq 2$ is the depth of S, which is tight.*

Proof.

We want to see that c, the number of connected components of $\bigcup Lay^D_i(S)$, is bounded by $(n-m+2)/2$ or, equivalently, that $n \geq 2c+m-2$. We use a charging scheme to prove this inequality.

If for some integer i the layer $Lay^D_{i+1}(S)$ has some connected components that are singletons, we know that each of these isolated vertices is enclosed by a cycle without chords of $Lay^D_i(S)$ (Lemma 3); we assign to each of these vertices a vertex of the corresponding enclosing cycle, which can be done in such a way that each of the single vertices gets assigned a different vertex. In order to see this, take the isolated vertices of $Lay^D_{i+1}(S)$ contained in a connected component C_i of $Lay^D_i(S)$. Considered on its own, the graph C_i is a cycle, possibly with some chords, and it is clear that we can assign to each of the bounded regions a different vertex in C_i.

On the other hand, if C_i has n_i vertices, the number of chords in C_i is at most $n_i - 3$ and the number of single vertices of $Lay^D_{i+1}(S)$ inside C_i is at most $n_i - 2$ (Lemma 2); therefore if we charge to each of these single vertices, besides itself, the vertex it steals from C_i, there are still 2 vertices left in C_i that we can charge to C_i itself. In this way, we see that two different vertices can be charged to every component in $\bigcup Lay^D_i(S)$, which gives a total of $2c$ vertices.

In addition, if the depth of S is m, by Proposition 5 we know that there exist at least $m-1$ nested cycles without chords, and that $m-2$ of them do not contain any component consisting of a single vertex. Therefore, there are at least $m-2$ connected components with three vertices or more and none of these vertices has been assigned to any isolated vertex. Hence we can pick one vertex from each of them that has not been charged to any component in the preceding scheme. In the overall, we have identified $2c+m-2$ distinct vertices in $\bigcup Lay^D_i(S)$, which yields $n \geq 2c+m-2$, as claimed.

The following example proves that the upper bound in the statement is tight.

Let us describe first the example for $m = 2$. Let S be the set $\{p\} \cup S_1 \cup S_2$ where p is the point $(0,1)$, $S_1 = \{(i,0) \mid i = 0, \cdots, \lfloor n/2 \rfloor\}$, and $S_2 = \{(i+1/2, \epsilon) \mid i = 0, \cdots, \lfloor n/2 \rfloor - 1\}$, where ϵ is small enough so that the circles tangent to the $+x$-axis in the points of S_1 passing through p, do not contain any of the points of S_2 (see Figure 9).

The Delaunay depth of p and all the points of S_1 is 1. All points in S_2 have depth equal to 2 and each one of them is contained in a different cycle of the layer $Lay^D_1(S)$ (see Figure 9).

Choose m such that $2 < m < n/3$. Start by putting $3(m-1)$ points in a sequence of nested triangles and one more point inside the innermost triangle. The triangles can be as far from each other as necessary. The rest of the points

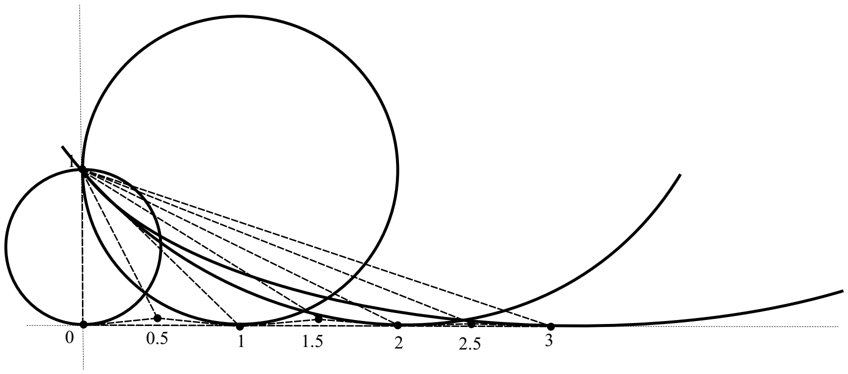

Fig. 9. An example with $n = 2k+2$ and $k = 2$. The union $\bigcup Lay_i^D(S)$ has $k+1 = 3$ connected components.

of S, at most $n - 3m + 2$, are distributed in pairs. Place each pair of points in any pair of contiguous layers: one point makes a new cycle in the layer where it is placed, and the other point is an isolated point in the new cycle. Notice that the pairs of points added in two contiguous layers form a distribution like Figure 9. In Figure 10, the $n - 3m + 2$ points all have been placed in the layers $Lay_1^D(S)$ and $Lay_2^D(S)$. $\square$

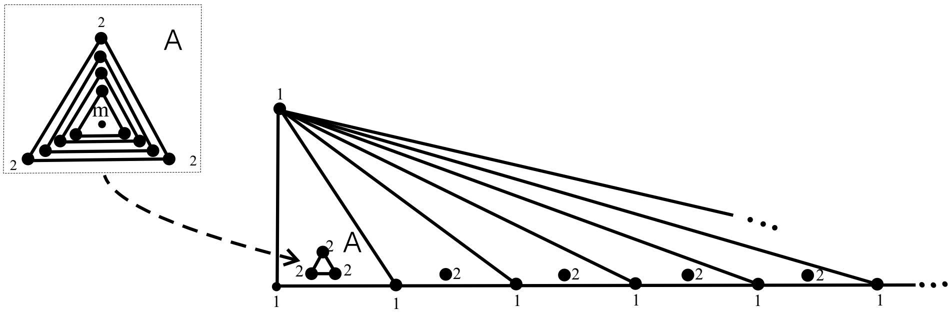

Fig. 10. On the left the connected component called A is enlarged. The set of points S has depth equal to m. The union $\bigcup Lay_i^D(S)$ has $\lfloor (n - m + 2)/2 \rfloor$ connected components.

Delaunay layers are not necessarily polygons; however, they form a structure composed of nested cycles of vertices of the same depth.

The depth of a point with respect to a set S depends on the Delaunay circles (i.e., circumcircles of Delaunay triangles) that contain the point, therefore the arrangement of Delaunay circles contains all the information about Delaunay levels (Figure 6). As the arrangement has size $O(n^2)$ and can be constructed in $O(n^2 \log n)$ time, one can obtain the Delaunay levels within this time. Nevertheless, in the following theorem we prove that in order to obtain all $Lev_i^D(S)$ it is not necessary to construct the whole arrangement of circles.

Remark 7 *Let C be a circle having exactly two points u and v of S on its boundary and containing no point of S in its interior. Then any circle crossing* *the two arcs determined by u and v in the boundary of C contains some interior point from S.*

Theorem 8 *Let S be a set of points in the plane and let its Delaunay depth be f. The union $\bigcup_{j\geq k} Lev_j^D(S)$, $k = 1, \ldots, f$ forms a sequence of sets nested by inclusion. The boundaries between $Lev_j^D(S)$ and $Lev_{j+1}^D(S)$, for $2 \leq j \leq f$, are curves composed by arcs of the Delaunay circles determined by two vertices u, v of $Lay_j^D(S)$ and one vertex w of $Lay_{j-1}^D(S)$.*

Proof. We proceed to determine the boundary between the consecutive levels of S, $Lev_j^D(S) = \{x \in \mathbb{R}^2 | \, d(x, S) = j\}$, and $Lev_{j+1}^D(S)$, for $2 \leq j \leq f$. For every point q of depth equal to j, with respect to a set S, there is at least one element $p \in S$ that is adjacent to q in $DG(S \cup \{q\})$ and has depth $j-1$ (in both $DT(S)$ and $DG(S \cup \{q\})$), and there cannot exist any empty circle through q and any point of S with depth less than $j-1$. Hence we can describe $Lev_j^D(S)$ as the union of all Delaunay circles that circumscribe a point of depth $j-1$ (that we denote by $\bigcup C_{j-1,-,-}$) minus the union of all Delaunay circles that circumscribe a point of depth less than $j-1$ (that we denote by $\bigcup C_{<j-1,-,-}$); that is,

$$Lev_j^D(S) = \bigcup C_{j-1,-,-} \setminus \bigcup C_{<j-1,-,-}.$$

Applying Proposition 3, which proves that for every point of depth j there is a cycle of $Lay_{j-1}^D(S)$ that contains it in its interior, we see that $Lev_j^D(S)$ is contained in the interior of the cycles of $Lay_{j-1}^D(S)$. Furthermore we also get the following properties: (a) If some layer, $Lay_i^D(S)$, has no cycles then there is no point for level $Lev_{i+1}^D(S)$ and the maximum level is $Lev_i^D(S)$; (b) the sets $\bigcup_{j\geq k} Lev_j^D(S)$, $k = 1, \ldots, f$ form a sequence of nested sets.

We find circles $C_{j,j,j-1} \in \bigcup C_{<j,-,-}$ intersecting the cycles of $Lay_j^D(S)$. The circles $C_{j,j,j-1}$ pass through pairs of vertices which are the endpoints of every non-chord edge of a cycle γ in $Lay_j^D(S)$ (see the cycle of $Lay_3^D(S)$ enclosing the dark region to the left of Figure 11).

These pairs of vertices divide the circle $C_{j,j,j-1}$ into two arcs: one exterior to the cycle γ, one interior. There may be other circles of $\bigcup C_{<j,-,-}$ that also cross the circle $C_{j,j,j-1}$, yet any circle of $\bigcup C_{<j,-,-}$ has in the boundary one vertex exterior to the cycle γ. Applying Observation 7, this circle of $\bigcup C_{<j,-,-}$ cannot cross both arcs of a circle $C_{j,j,j-1}$.

Therefore the boundary between $Lev_j^D(S)$ and $Lev_{j+1}^D(S)$ is only determined by the arcs of the circles $C_{j,j,j-1}$ (see Figure 12 for an illustration). □

Theorem 8 proves that the overall size of the Delaunay levels is $O(n)$ and

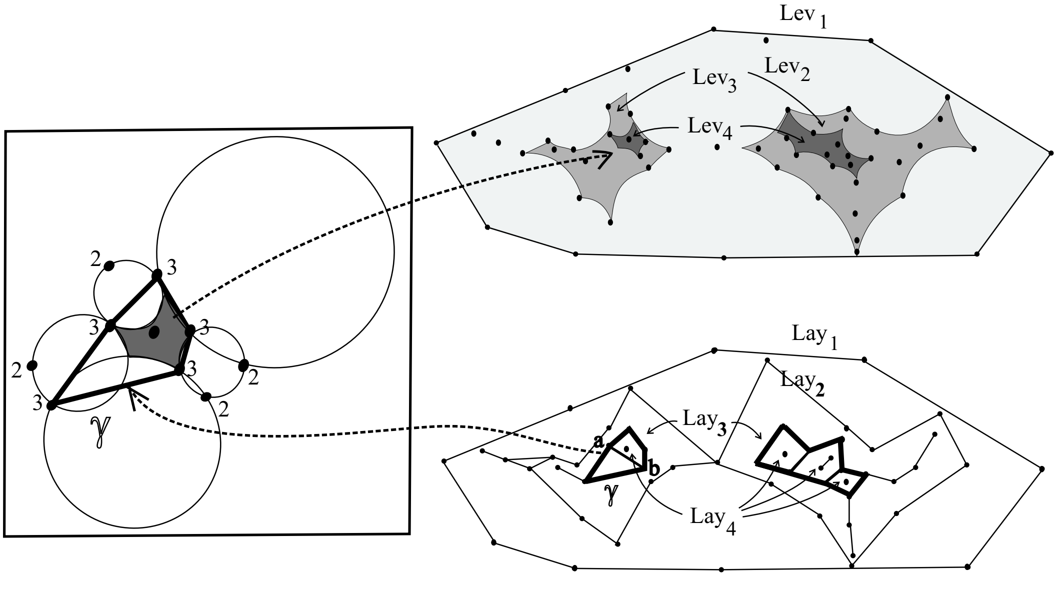

Fig. 11. The Delaunay circles $C_{3,3,2}$ defined by two vertices of $Lay_3^D(S)$ and one vertex of $Lay_2^D(S)$, determine the boundary between $Lev_3^D(S)$ and $Lev_4^D(S)$, which consists of the inner boundary of the union of $C_{3,3,2}$. Notice that chord ab has been "discarded" as useless for obtaining the level.

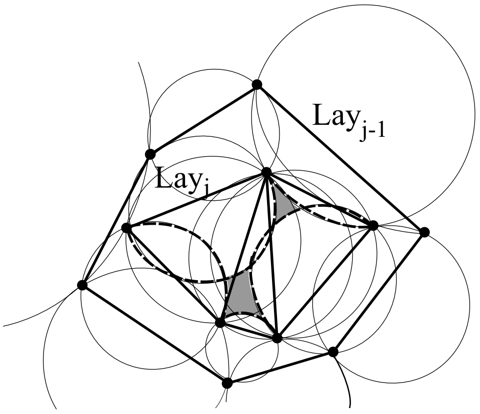

Fig. 12. The shaded region is $Lev_{j+1}^D(S)$.

justifies the steps of the following algorithm.

Algorithm 1 COMPUTATION OF DELAUNAY DEPTH CONTOURS OF S, DELAUNAY LEVELS.

INPUT: *Set of points S in non-degenerate position.*

OUTPUT: *Delaunay depth contours of S.*

1. *Compute $DT(S)$.*
2. *Compute the Delaunay depths for all points in S.*
3. *Compute the boundaries of the levels as follows: $Lev_1^D(S)$ is the convex hull of S; for every $j \geq 2$, construct the inner boundary of the union of*

Delaunay circles $C_{j,j,j-1}$ *defined by two vertices* u, v *of* $Lay_j^D(S)$ *and one vertex* w *of* $Lay_{j-1}^D(S)$ *(Figure 12).*

$DT(S)$ can be computed in $O(n \log n)$ time and Step 2 takes $O(n)$ additional time. Every boundary in Step 3 can be computed in $O(t \log^2 t)$ time, where t is the number of Delaunay circles $C_{j,j,j-1}$ considered in the currently computed layer, by using the algorithm described in [4](p. 97). Taking into account that the total number of Delaunay circles is $O(n)$, Step 3 takes $O(n \log^2 n)$ global time, which is also the overall time for the algorithm. Notice that the expected time for Step 3 is $O(n \log n)$ [4], and therefore, the expected running time for the entire algorithm is $O(n \log n)$.

Algorithm 1 computes all levels of S in $O(n \log^2 n)$ time. Therefore it also yields the *Delaunay median* in this time. In Figure 13 we see an illustration where the inner level, $Lev_6^D(S)$, has two connected components: the centroids of each one of these regions are the Delaunay medians of S.

As a consequence of the preceding paragraphs we can state the following theorem.

Theorem 9 *The Delaunay levels of a set of* n *points in the plane can be constructed within* $O(n \log^2 n)$ *time in the worst case and in* $O(n \log n)$ *expected time.*

It is also natural to consider how strong the change in the Delaunay depths of a point set can be after the insertion of a new point. This is the issue we study next.

Proposition 10 *The insertion of a new point* p *in a set of* n *points* S*, such that both* S *and* $S \cup \{p\}$ *are in non-degenerate position, can change the depth of a point* $q \in S$ *by at most* $\lfloor n/3 \rfloor - 2$*, which is tight, and the depth of the set can change by* $\lfloor n/3 \rfloor - 3$*, which is tight up to one unit.*

Proof. When a new point p is inserted, the depth of a point $q \in S$ may change because p becomes a new neighbor of q, or even if its neighbors are unchanged, they get new depths. If q is a convex hull vertex its depth can change from 1 to at most 2; if q is an interior point, its depth can change from at most $n/3$ (the maximum possible depth) to 2. This value is also an upper bound for the change of the depth of the whole set.

Let us see now an example of a point set S with depth $n/3$, in which the insertion of a suitable point p modifies the depth of a certain point in S from $n/3$ to 2, and such that the depth of $S \cup \{p\} = 3$ while the depth of S is $n/3$. Consider two triangles homothetic from their common circumcenter such that the circumcircle C of the inner triangle T_{int} crosses each edge of the outer

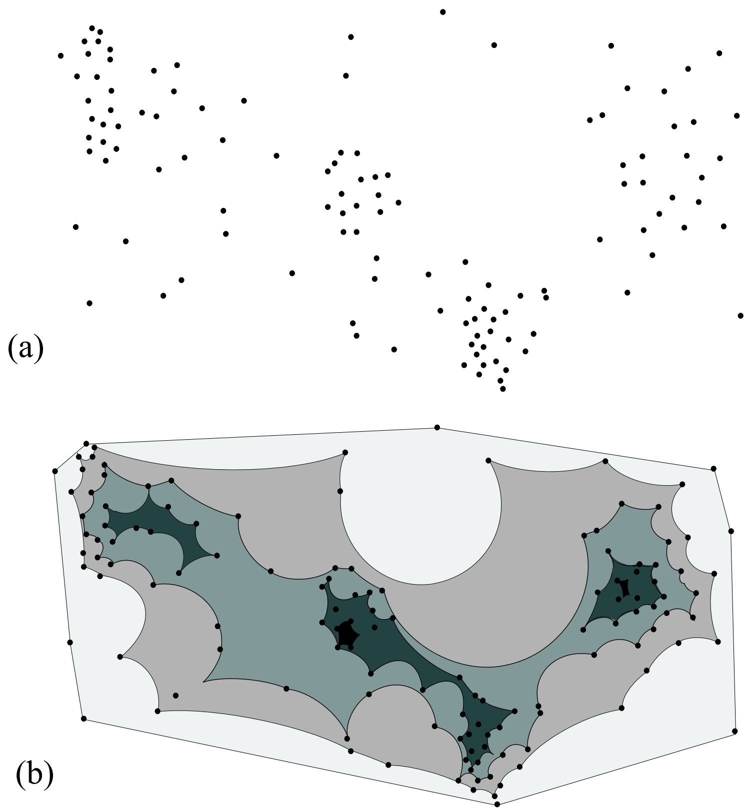

Fig. 13. (a) A point set S. (b) Levels of S. The boundaries of the levels are the *Delaunay depth contours.*

triangle T_{ext} twice; see Figure 14. Then S is defined by taking the six vertices of the triangles and evenly placing points in the segments s_1, s_2 and s_3 that join corresponding vertices of both triangles; these new points are slightly perturbed inside the segments in such a way that S is in non-degenenerate position. Notice that the interior of the disk bounded by C is empty of points of S and that part of it is outside $CH(S)$. The Delaunay layers of S are triangles and the depth of S is $n/3$; layers and levels are shown in Figures 15 (a) and (b).

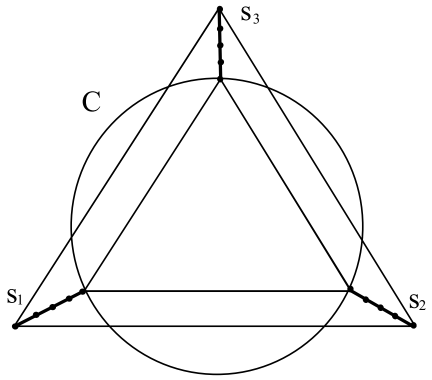

Fig. 14. The points of S lie on the segments $s1$, s_2 and s_3.

We now insert a point p (refer to Figure 15) that is exterior to $CH(S)$ and interior to the disk bounded by C such that $S \cup \{p\}$ is in non-degenerate position. In this way, p is adjacent to the three vertices of T_{int} and to all points placed on the two closest segments, say s_1 and s_2. Hence p is adjacent to points of depth $n/3$ in S (the vertices of T_{int}) and to points of depth 1 (the vertices of T_{ext}).

Let us consider the depths in $S \cup \{p\}$. The point p has depth 1 (it is exterior to $CH(S)$) and any of its neighbors that is not in $CH(S \cup \{p\})$ now has depth 2. Therefore, the three vertices of T_{int} that had depth $n/3$ in S, get depth 2 in $S \cup \{p\}$, a change as claimed.

The points of depth 1 and 2 in S still have the same depth in $S \cup \{p\}$. The edges of $DT(S \cup \{p\})$ with an endpoint in s_3 are the same as in $DT(S)$; only edges between s_1 and s_2 have changed. As a consequence, the vertex of $Lay_2^D(S)$ from s_3 and the neighbors of p in $Lay_2^D(S \cup \{p\})$ determine a cycle of $Lay_2^D(S \cup \{p\})$ (Figure 15, (c)). The other points that remain on s_3 are of depth 3. Therefore, after the insertion of p, the depth of S changes from $n/3$ to 3. $\square$

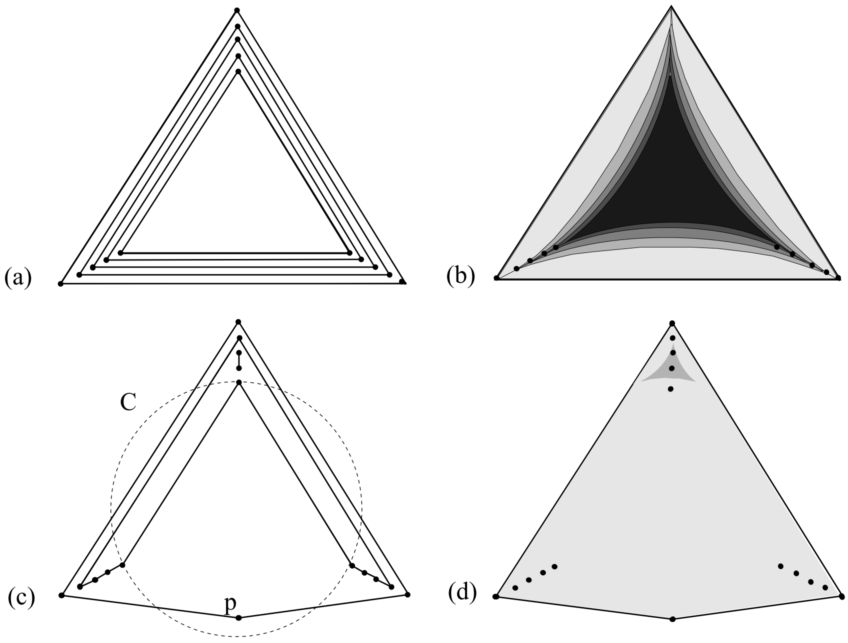

Fig. 15. (a) Delaunay layers of S. (b) Delaunay levels of S. (c) Delaunay layers of $S \cup \{p\}$. (d) Delaunay levels of $S \cup \{p\}$.

4 Computing Delaunay depth

As mentioned in the introduction, several notions of the depth of a point p with respect to a data set $S = \{s_1, \ldots, s_n\}$ have been considered, and the computation of the depth of p is a problem that has received much attention. When S and p are the entry data, the Tukey depth of p, its simplicial depth, and its Oja depth can be computed in $O(n \log n)$ time [33]. Aloupis et al [2] proved that this bound is also a tight for the first two cases (Tukey and simplicial depth) and recently an identical result has been proved for Oja depth [3].

The convex depth of p can be computed in $O(n \log n)$ time, since it suffices to find the layers of $S \cup \{p\}$, and it is easy to see that this bound is tight. The Delaunay depth can also be found in $O(n \log n)$ time, since it suffices to build $DG(S \cup \{p\})$ which is done in $O(n \log n)$ time (remember that $DG(S \cup \{p\})$ is a linear size structure because S is in non-degenerate position) and then find the depth of p in additional $O(n)$ time.

We will next show that this bound is tight.

We use a reduction from integer element uniqueness. Yao [40] proved an $\Omega(n \log n)$ lower bound for this problem under the algebraic computation tree model. We are given a set of integers $M = \{x_1, \ldots, x_n\}$ as input to the integer element uniqueness problem. Assume without loss of generality that each x_i is positive and that the minimum and the maximum of the given integers (which can be computed in linear time) are x_1 and x_n, respectively. Now we change the input to the integer element uniqueness problem to be the set $M' = \{y_1 = n^2 x_1, y_2 = n^2 x_2, \ldots, y_n = n^2 x_n\}$. Clearly all elements in M are different if and only if all elements in M' are different. We have

$$n^2 \leq y_1 = \min\{y_1, \ldots, y_n\} \leq y_n = \max\{y_1, \ldots, y_n\}.$$

On the other hand, notice that, by construction, we have $|y_i - y_j| \geq n^2$ whenever $y_i \neq y_j$.

As a first consideration, notice that given any two different positive integers, α and β, the four points $p = (\alpha, 0)$, $q = (\beta, 0)$, $p' = (0, \alpha)$ and $q' = (0, \beta)$ (refer to Figure 16) are in degenerate position from the Delaunay triangulation viewpoint because in the quadrilateral $pqq'p'$ the angles at p and q' add up to 180° and therefore the four points are cocircular. Nevertheless if we translate down by an infinitesimal amount ϵ the points p' and q' to p'' and q'', then these angles increase strictly and the quadrilateral $pqq''p''$ has a uniquely defined Delaunay triangulation, namely the boundary of the quadrilateral together with the diagonal $q''p$. Observe also that if we perturb the vertices of $pqq''p''$ inside a very small neighborhood, say a disk of radius ϵ^2 (which is an infinitesimal quantity with respect to ϵ), the Delaunay triangulation of the new point set is still the same.

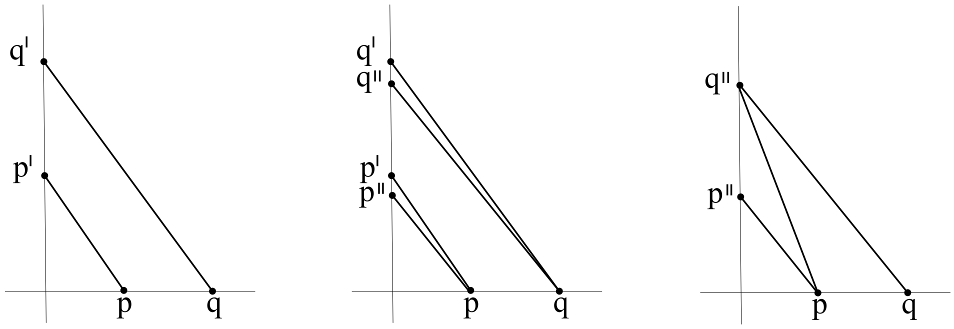

Fig. 16. The points p' and q' are translated to p'' and q'' and the vertices of the quadrilateral $pqq''p''$ are possibly perturbed.

The intuition for the construction we develop next comes from the following idea: If $z_1, \ldots, z_n$ are *distinct* positive integer numbers, consider for each z_i the four points

$$z_i^{(1)} = (z_i, 0),\ z_i^{(2)} = (0, z_i - \epsilon),\ z_i^{(3)} = (-z_i, 0),\ z_i^{(4)} = (0, -z_i + \epsilon).$$

Then we know from the preceding paragraph that the Delaunay triangulation of the point set $Z = \{(0,0)\} \cup (\cup_{i,j}\{z_i^{(j)}\})$ is as shown in Figure 17 and that the Delaunay depth of the origin $(0,0)$ in Z is exactly $n+1$. We would like to build a similar construction for the distinct numbers in the set $M' = \{y_1, y_2, \ldots, y_n\}$ in such a way that those that are repeated have no influence on the depth of the origin, and this number would detect how many numbers in M' are different.

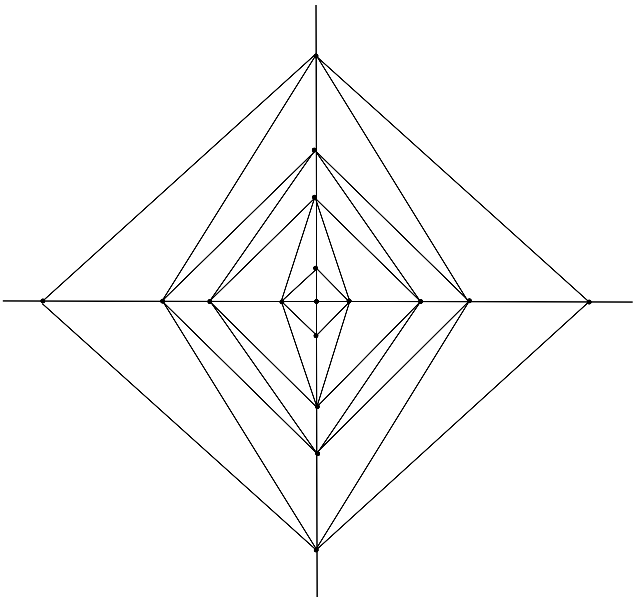

Fig. 17. Set of points Z and its Delaunay traingulation.

We are ready now for describing the precise construction; besides the basic idea mentioned above, several technical details are required to ensure that the resulting point set is in non-degenerate position.

Let ϵ be the number $\epsilon = \frac{1}{y_n^2 \cdot n^2}$. For each y_i consider the following points:

$$a_i = (y_i, i\epsilon^2), \quad a_i' = (y_i - i, 0), \quad b_i = (-i\epsilon^2, y_i), \quad b_i' = (0, y_i - i - \epsilon),$$

$$c_i = (-y_i, -i\epsilon^2), \quad c_i' = (-y_i + i, 0), \qquad d_i = (i\epsilon^2, -y_i), \quad d_i' = (0, -y_i + i + \epsilon).$$

The points a_i' lie on the $+x$-axis and the points a_i are close to it. More precisely, a_i is the point $(y_i, 0)$ lifted in the $+y$ direction by an extremely small amount that depends on i, therefore if $y_i = y_j$ the points a_i and a_j are different but extremely close and lie on the same vertical. Observe that we cannot have $a_i' = a_j'$ when $i \neq j$, because in the case $y_i = y_j$ the equality $y_i - i = y_j - j$ would yield $i = j$, against the hypothesis, and in the case $y_i \neq y_j$ the equality $y_i - i = y_j - j$ would give the equality $|y_i - y_j| = |i - j|$, which is impossible because $|y_i - y_j| \geq n^2$. Therefore we have exactly $2n$ different points taking together all the a_i and a_i'.

Applying a rotation of 90° around the origin to the points a_i and a_i', followed by a vertical translation with vector $(0, -\epsilon)$ we obtain the points b_i and b_i', respectively. The points d_i and d_i' are similarly obtained from a_i and a_i', but the angle of rotation is 270° and the vector of translation is $(0, -\epsilon)$. Finally, the points c_i, and c_i' are respectively images of a_i and a_i' after a rotation of 180° around the origin.

We denote by S the union of all the points a_i, a_i', b_i, b_i', c_i, c_i', d_i and d_i'. Any set of equal numbers contained in M' produces a set of points of the type a_i that are covertical and extremely close. Let u and v respectively be the lowest and highest point among these points and let u' and v' be the corresponding points of type b_i after a rotation of 90° and a translation of $(0, -\epsilon)$. It is clear that u' and v are neighbors in $DT(S)$ (Figure 18).

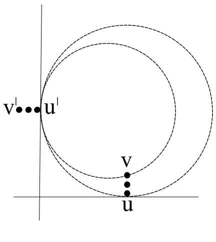

Fig. 18. The points u' and v are neighbors in $DT(S)$.

On the other hand, the (closed) circle of radius $r = y_i - i$ centered at $(y_i - i, y_i - i)$ covers a_i' and b_i'; let us prove that no point of type a_j is inside that circle, then it follows that a_i' and b_i' are neighbors in $DT(S)$. This requires only some elementary computations based on our choice of infinitesimals. The maximum ordinate of any point a_j is at most

$$n\epsilon^2 = n \cdot \frac{1}{y_n^4 \cdot n^4} = \frac{1}{y_n^4 \cdot n^3};$$

the maximum value of the radius r is y_n and the smallest possible length of a vertical segment between the circle and a point on the x-axis with integer abscissa, different from the contact point of the circle and the axis, is the value $t = y_n - \sqrt{y_n^2 - 1}$ (Figure 19). Therefore it is enough to show that

$$t = y_n - \sqrt{y_n^2 - 1} > \frac{1}{y_n^4 \cdot n^3},$$

which is true because

$$y_n - \sqrt{y_n^2 - 1} > \frac{1}{y_n^2} > \frac{1}{y_n^4 \cdot n^3}.$$

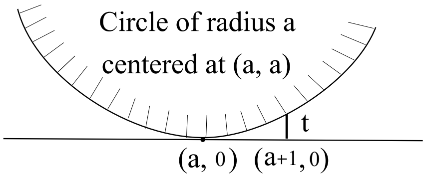

Fig. 19. For large values of a the value of t is greater than $1/a^2$, i.e., $t = a - \sqrt{a^2 - 1} > \frac{1}{a^2}$.

Finally, notice that if we have a set S_1 of covertical points of type a_i and another set S_2 of covertical points also of the type a_i, such that their abscissae are consecutive as different integers in M', then the Delaunay triangulation $DT(S)$ on their vicinity is as shown in Figure 20.

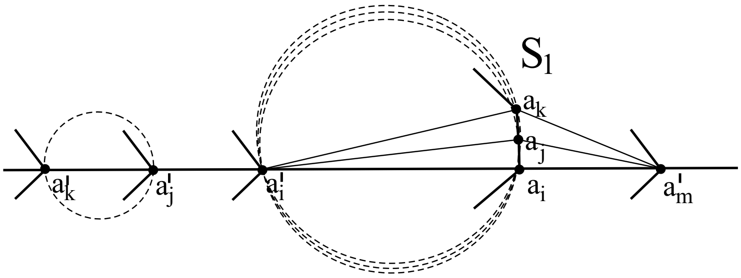

Fig. 20. Delaunay triangulation in the vicinity of the $+x$-axis.

From the preceding considerations we see that the Delaunay triangulations $DT(S)$ and $DT(S \cup \{p\})$ are well defined; an approximate drawing of the latter for a small example is shown in Figure 21. A shortest path in $DT(S \cup \{p\})$ from p to the convex hull goes through all the points a_i' and one point from

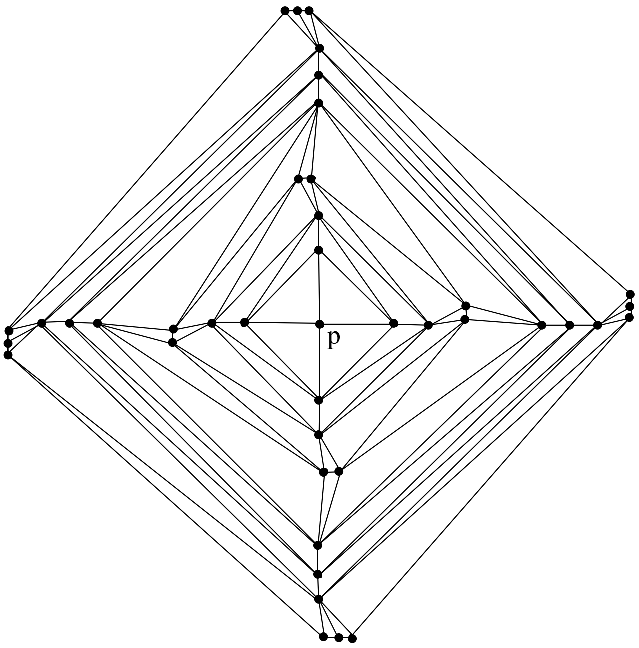

Fig. 21. Set of points S and Delaunay triangulation of $S \cup \{p\}$.

each set of covertical points a_i. Therefore the depth of p in $S \cup \{p\}$ is exactly $2+n+$(number of distinct numbers in M), and equals $2n+2$ if and only if all the elements of M are distinct. This completes the proof of the claimed lower bound. The following result has thus been established:

Theorem 11 *Computing the Delaunay depth of a point p with respect to a data set* $S = \{s_1, \cdots, s_n\}$ *requires* $\Omega(n \log n)$ *time in the worst case, in the algebraic decision tree model, which matches the upper bound complexity of the algorithms for its computation.*

If we allow an additional preprocessing step to the given point set, we have different alternatives for computing the level of a new point. For example, the preprocessing might consist of computing the Delaunay triangulation, or even the arrangement of the Delaunay circles. Nevertheless, the most natural approach is to compute the Delaunay levels in a first step (which requires $O(n \log^2 n)$ time), as this gives a plane subdivision of size $O(n)$; standard point-location methods can then be used [19]. In particular, the approach of Sarnak and Tarjan [35] can be easily adapted and allows $O(\log n)$ query time.

5 Concluding remarks

In this work we have studied the Delaunay depth function, the stratification that this depth induces in the point set (layers) and in the whole plane (levels). We have developed algorithms for computing the *Delaunay depth contours* and the depth of any query point with respect to a given point set. The stratification suggests that Delaunay depth may be more suitable than others for cluster detection and visualization.

There are two issues that deserve specific comment. On one hand, we have assumed throughout the paper that the point sets S given as input data contain no degeneracies, in the sense made precise in Section 2. There are several options when those degeneracies are allowed. Maybe the simplest is to use any of the perturbation schemes that have been proposed and lead to non-degenerate sets (for a review of perturbation schemes, see [11], [41] and [12]). Another possibility when there are four or more cocircular points having no other point interior to the circle passing through them is to use some uniquely determined triangulation that completes the Delaunay edges already accepted without any ambiguity due to cocircularity; an example of this is the *globally equiangular triangulation* defined by Edelsbrunner [10], which can be computed in $O(n \log n)$ time [24]. Finally, a third possibility is to consider the *Delaunay graph* of S; that is, the graph having the points in S as vertices, two points p and q being adjacent when there is some circle through p and q whose interior is empty of points from S. We recall that in this case a set of cocircular points from S such that the circle through them contains no other point in S would define a complete subgraph of the Delaunay graph, that hence may have quadratic size. For this third option all results and algorithms in this paper should be reconsidered from scratch.

On the other hand it is worth noticing that measuring the depth of a query point $p \notin S$ inside $DG(S \cup \{p\})$ is somehow a shortcoming of this notion of depth, because adding a point to S can drastically change the structure of the depth layers, yet the same consideration applies to any notion of proximity graph depth, for which the depth definitions that have been given are totally analogous [16], [17], [28], [29], [30] and [31]. This raises as a new direction of research to study some alternative definitions providing more robustness. For example, a second Delaunay depth definition for a point $p \notin S$ could be defined by assigning to p the depth of the closest vertex of the Delaunay triangle that p is located in. One could also use barycentric coordinates and combine the three depths of the corners in order to get a continuous depth notion, as suggested in another situation by Tukey [14] for rational depths. We plan to explore these alternatives in the near future. Nevertheless, notice that none of these alternative definitions can overcome the fact that the depth structure in a proximity graph is weak regarding insertions and deletions, and therefore it is not a dynamically robust concept.

As for specific open problems, let us mention that we do not know whether a Delaunay median (a point of maximum Delaunay depth) can be computed directly, in other words, avoiding the depth computation for the whole point set.

Acknowledgments

We would like to thank the anonymous referees for many useful remarks and suggestions. We also thank Alfredo de las Vegas, who developed the software for the computation of Delaunay levels and layers; this software is available at http://www.dma.fi.upm.es/mabellanas/delonedepth.

References

[1] Aloupis, G. Geometric Measures of Data Depth. In: Data Depth: Robust Multivariate Analysis, Computational Geometry and Applications, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, to appear.

[2] Aloupis, G. Cortés, C. Gómez, F. Soss, M. and Toussaint, G., 2002. Lower bounds for computing statistical depth. Computational Statistics & Data Analysis 40 223–229.

[3] Aloupis, G. and McLeish, E., 2005. A Lower Bound for Computing Oja Depth. Information Processing Letters 96(4) 151–153.

[4] Agarwal, P. and Sharir, M., 2000. Arrangements and its applications. In: J.R. Sack and J. Urrutia (Ed.), Handbook of Comput. Geom., North-Holland.

[5] Barnett, V., 1976. The ordering of multivariate data (with discussion). J. Roy. Statist. Soc. Ser., A 139(3) 318–352.

[6] Bai, Z.D. and He, X., 1999. Asymptotic distributions of the maximal depth estimators for regression and multivariate location. Ann. Statist., 27 1616–1637.

[7] Chazelle, B., 1985. On the Convex Layers of a Planar Set, IEEE Transactions on Information Theory, IT-31 4 509–517.

[8] Claverol, M., 2004. Geometric Problems on Computational Morphology. Ph. D Thesis, Universitat Politècnica de Catalunya, Barcelona, Spain.

[9] Cole, R. Sharir, M. and Yap, C.K., 1987. On K-hulls and related problems. SIAM Journal of Computing, 16 61–77.

[10] Edelsbrunner, H., 1987. Algorithms in Combinatorial Geometry. Springer -Verlag, New York, USA.

[11] Emiris, I. and Canny, J., 1991. A general approach to removing degeneracies. In: Proc. 32nd Ann. IEEE Symp. Foundations of Computer Science, 405–413.

[12] Emiris, I. Z. Canny, J. F. and Seidel, R., 1997. Efficient perturbations for handling geometric degeneracies. Algorithmica, 19(1–2) 219–242.

[13] Eppstein, D., 1997. Fast hierarchical clustering and other applications of dynamic closest pairs. In: Proc. 9th Annu. ACM-SIAM Sympos. Discrete Algorithms, New Orleans, 131–138.

[14] Green, P.J., 1981. Peeling Bivariate Data. In: V. Barnett (Ed.), Interpreting multivariate data, Wiley.

[15] Hubert, P.J., 1972. Robust statistics: a review. Ann. Math. Statistics, 43(3) 1041–1067.

[16] Hugg, J. Rafalin, R. Seyboth, K. and Souvaine, D., 2006. An experimental study of old and new depth measures. In: Proc. of the 8th International Workshop on Algorithm Enginneering and Experiments (ALENEX).

[17] Hugg, J. Rafalin, E. and Souvaine, D., 2005. Depth Explorer - A Software Tool for Analysis of Depth Measures. Tufts CS Technical Report 2005-06, Tufts University, November 2005. Abstract appeared in: Proc. of the 15th Annual Fall Workshop on Computational Geometry, UPenn.

[18] Jaromczyk, J.W. and Toussaint, G.T., September, 1992. Relative neighborhood graphs and their relatives. In: Proc. IEEE, vol. 80(9) 1502–1517.

[19] Kirkpatrick, D.G., 1983. Optimal search in planar subdivisions. SIAM J. of Computing, 12 28–35.

[20] Liu, R.Y., 1990. On a notion of data depth based on random simplices. Ann. Statist., 18 405–414.

[21] Liu, R.Y. Paralelius, J. and Singh, K., 1999. Multivariate analysis by data depth: descriptive statistics, graphics and inference. Ann. Statist., 27 783–840.

[22] Matousek, J., 1991. Computing the center of planar point sets. In: Computational Geometry: Papers from the DIMACS Special Year, J. E. Goodman, R. Pollack, and W. Steiger, editors, American Mathematical Society, Providence, 221–230.

[23] Miller, H.K. Ramaswami, S. Rousseeuw, P. Sellares, T. Souvaine, D. Streinu, I. and Struyf, A., 2003. Efficient computation of location depth contours by methods of Computational Geometry. Statistics and Computing.

[24] Mount, D.M. and Saalfed, A., 1988. Globally-Equiangular Triangulations of Co-Circular Points in $0(n \log n)$ Time. In: Proc. of the 4th Annual Sympos. on Computational Geometry, ACM Press, 143–152.

[25] Nanopoulos, A. Theodoridis, Y. and Manolopoulos, Y., 2001. C^2P: Clustering based on Closest Pairs. In: Proc. of the 27th VLDB Conference, Roma.

[26] Oja, H. 1983. Descriptive statistics for multivariate distributions. Statist. Probab. Lett., 1 327–332.

[27] Okabe, A. Boots, B. and Sugihara, K., 1992. Spatial Tessellations: Concepts and Applications of Voronoi Diagrams. Wiley, Chichester, 362–363.

[28] Rafalin, E. and Souvaine, D. Data Depth Contours - a Computational Geometry Perspective. Tufts CS Technical Report, 2004-01. Tufs University, 2004.

[29] Rafalin, E. and Souvaine, D., 2004. Computational Geometry and Statistical Depth Measures, Theory and Applications of Recent Robust Methods, edited by M. Hubert, G. Pison, A. Struyf and S. Van Aelst, In Series: Statistics for Industry and Technology, Birkhauser, Basel, 283–296.

[30] Rafalin, R. Seyboth, K. and Souvaine, D., 2005. Path Length in Proximity Graphs as a Data Depth Measure. Tufts CS Technical Report 2005-05,

Tufts University, November 2005. Abstract appeared in: Proc. of the 15th Annual Fall Workshop on Computational Geometry, UPenn.

[31] Rafalin, R. Seyboth, K. and Souvaine, D. Proximity graph depth, depth contours, and a new multimodal median. Submitted for publication, 2006.

[32] Rousseeuw, P.J. and Hubert, M., 1999. Regression depth. J. Amer. Statist. Assoc., 94 388–402.

[33] Rousseeuw, P.J. Ruts, I., 1996. Algorithm AS 307: bivariate location depth. J. Roy. Statist. Soc. Ser. C, 45 516–526.

[34] Rousseeuw, P.J. Struyf, A., 2004. Computation of robust statistics: depth, median, and related measures. In: Goodman and J. O'Rourke (Ed.), Handbook of Discrete and Comput. Geom., J.E., Discrete Mathematics and Its Applications.

[35] Sarnak, N. Tarjan, R. E., 1986. Planar point location using persistent search trees. Communications of the ACM, 29(7) 669–679.

[36] Sibson, R., 1981. A brief description of natural neighbour interpolation. In: V. Barnett (Ed.), Interpolating multivariate data, Wiley, Chichester, 21–36.

[37] Small, C.G., 1990. A survey of multidimensional medians. Internat. Statistical Review, 58 263–277.

[38] Struyf, A. Hubert, M. and Rousseeuw, P.J., 1997. Integratint robust clustering techniques in S-PLUS. Comput Statist. Data Anal., 26 17–37.

[39] Tukey, J.W., 1975. Mathematics and the picturing of data. Proc. of the Intern. Congress of Mathematicians, 2 523–531.

[40] Yao, A.C.C., 1989. Lower bounds for algebraic computation trees with integer inputs. In: Proc. 30th Annual IEEE Sympos. on Foundations of Computer Science, 308–313.

[41] Yap, C., 1990. Symbolic treatment of geometric degeneracies. Journal of Symbolic Computation, 10 349–370.

[42] Zuo, Y. Serfling, R., 2000. General notions of statistical depth functions. Ann. Statist., 28 461–482.