INTERSECTION OF LINE-SEGMENTS

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INTERSECTION OF LINE-SEGMENTS

Problem

Input: \( n \) line-segments in the plane, \( s_i = (p_i, q_i), \ i = 1 \ldots n \).
Output: the \( k = O(n^2) \) intersections of line-segment pairs, \((x, y, i, j)\).
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Some applications (in addition to those that you already know)

**Geographic information systems**
Detecting the intersections among the elements of the different layers of information (cities, roads, services, ...)

**Realistic visualization**
Eliminating the hidden portions of a scene
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Brute force solution

Check the intersection of the \( \binom{n}{2} \) pairs of line-segments. This algorithm runs in \( \Theta(n^2) \) time.
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Problem complexity

The problem has complexity $\Omega(n^2)$, because there exist line-segment configurations with $\binom{n}{2}$ intersections.
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Nevertheless, there exist sets of $n$ line-segments with a number of intersections substantially smaller than $\binom{n}{2}$. 
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The problem has complexity \( \Omega(n^2) \), because there exist line-segment configurations with \( \binom{n}{2} \) intersections.

Nevertheless, there exist sets of \( n \) line-segments with a number of intersections substantially smaller than \( \binom{n}{2} \).

Output-sensitive solution

Algorithm whose running time depends on the number of intersections to be reported.
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Solution

*Sweep line algorithm*
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Solution

**Sweep line algorithm**

A straight line (vertical, in this case) scans the object (the line segments) and allows detecting and computing the desired elements (intersection points), leaving the problem solved behind it. The sweeping process is discretized.
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**Input:** $n$ line-segments in the plane, $s_i = (p_i, q_i)$, $i = 1 \ldots n$.
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Essential elements of a sweep line algorithm:

- **Sweep line**
  Data structure storing the information of the objects intersected by the sweeping line, sorted along the line.

- **Events queue**
  Priority queue keeping the information of the algorithm stops, i.e., all the positions where the sweep line structure changes.
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Observation 1

If two line-segments have disjoint projections onto a given line, then they are disjoint.
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Observation 2

When sweeping a set of line-segments with a line, two intersecting line-segments need to be consecutive in the sweeping line right before their intersection point.
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When sweeping a set of line-segments with a line, two intersecting line-segments need to be consecutive in the sweeping line right before their intersection point.
Bentley-Ottman’s algorithm

This is a sweep-line algorithm.
Bentley-Ottman’s algorithm

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Hypothesis (to be eliminated later on)

1. There are no repeated abscissae, i.e.: there are no vertical line-segments, and no two endpoints of two line-segments, no two intersection points of line-segments, no endpoint and intersection point, lie in the same vertical line.

2. At each intersection point, only two line-segments intersect.
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**Sweep line**

Stabbed line-segments, in vertical order.
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**Sweep line**

Stabbed line-segments, in vertical order.

**Events**

- All endpoints of the line-segments (known a priori).
- All intersection points of line-segments (found on the fly).
Bentley-Ottman’s algorithm
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**Initialization:**
- Sort the \( 2n \) endpoints by abscissa and store the information in \( E \).
- Line \( L \) starts empty.
**Advance**

While $E \neq \emptyset$ do:

1. $p = \min E$

2. If $p = \text{start}(s)$, then:
   - Insert $s$ in $L$
   - If $s^-$ and $s$ intersect to the right of $p$, insert their intersection point in $E$ and report it (if needed). Do the same for $s^+$. 

3. If $p = \text{end}(s)$, then:
   - If $s^-$ and $s^+$ intersect to the right of $p$, insert their intersection point in $E$ and report it (if needed).
   - Delete $s$ from $L$

4. If $p = s_1 \cap s_2$ with $s_1 <_L s_2$, then:
   - If $s_1^-$ and $s_2$ intersect to the right of $p$, insert their intersection point in $E$ and report it (if needed). Do the same for $s_2^+$ and $s_1$.
   - Transpose $s_1$ and $s_2$ in $L$

5. Delete $p$ from $E$
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Bentley-Ottman’s algorithm: simulation
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\[q_2\quad p_6\quad q_6\quad r_{15}\quad q_5\quad q_1\]

\[r_{23}\quad r_{12}\quad r_{13}\quad r_{15}\]
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Correctness

• The algorithm finds all intersections (due to Observation 2).

• The algorithm does not find any faked intersection (all intersections are checked).
Bentley-Ottman’s Algorithm

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- The algorithm does not find any faked intersection (all intersections are checked).

Dealing with degenerate cases

- In order to deal with input data containing more than one point sharing the same abscissa, the event queue $E$ must store the points in lexicographical order (and not only by abscissae).
- The algorithm can trivially detect whether two or more line-segments intersect in more than one point (i.e., intersect in a line-segment), since it stops at their endpoints.
- A slight modification also allows to deal with input data in which three or more line-segments intersect at the same point: in this case, the algorithm inverts their order in the sweep line at the intersection point event.
Bentley-Ottman’s Algorithm

Data structures
Bentley-Ottman’s Algorithm

Data structures

Sweep line, $L$:
- Keeps the total order of the stabbed line-segments and supports:
  - $\text{insert}(s)$
  - $\text{delete}(s)$
  - $\text{transpose}(s_1, s_2)$
  - $\text{previous}(s)$
  - $\text{next}(s)$

A dictionary (balanced binary tree) allows to perform each of these operations in $O(\log n)$ time.
Bentley-Ottman’s Algorithm

**Data structures**

Events queue, $E$:

Keeps the total order of the events and supports:

- minimum (report and extract)
- insert($p$)
- memberQ($p$)

A priority queue (balanced binary tree) allows to perform each of these operations in $O(\log n)$ time.
Bentley-Ottman’s Algorithm

Complexity (time)
Bentley-Ottman’s Algorithm

Complexity (time)

Initialization: $O(n \log n)$

Advance (performed $2n + k$ times):
1. $O(\log n)$
2. $O(\log n)$
3. $O(\log n)$
4. $O(\log n)$

Total: $O((n + k) \log n)$

Initialization:
- Sort the $2n$ endpoints by abscissa and store the information in $E$.
- Line $L$ starts empty.

Advance
While $E \neq \emptyset$ do:
1. $p = \min E$
2. If $p = \text{start}(s)$, then...
3. If $p = \text{end}(s)$, then...
4. If $p = s_1 \cap s_2$ with $s_1 <_L s_2$, then...
5. Delete $p$ from $E$
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Advance (performed $2n + k$ times):
1. $O(\log n)$
2. 3. 4. $O(\log n)$
5. $O(\log n)$

Total: $O((n + k) \log n)$

The previous counting corresponds to the non degenerated case. When each intersection point, $v_i$, may correspond to more than two intersecting line-segments, the total running time of the advance step of the algorithm is $O((\sum_{i=1}^{k} \text{degree}(v_i)) \log n)$. However, considering the points $v_i$ as vertices of the graph:

$$\sum_{i=1}^{k} \text{degree}(v_i) \leq 2e = O(e) = O(v) = O(2n + k) = O(n + k).$$
Bentley-Ottman’s Algorithm

Complexity (space)
Bentley-Ottman’s Algorithm

Complexity (space)

At each step of the algorithm, the sweep line stores at most \( n \) line-segments.
Bentley-Ottman’s Algorithm

Complexity (space)

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In the formulation exposed so far, the event queue may at some point store all intersection points, which are $k = O(n^2)$. 
Bentley-Ottman’s Algorithm

Complexity (space)

At each step of the algorithm, the sweep line stores at most \( n \) line-segments.

In the formulation exposed so far, the event queue may at some point store all intersection points, which are \( k = O(n^2) \).

However, a slight modification allows the event queue to store, at each step of the algorithm, at most \( n - 1 \) intersection events. This can be achieved if, at each step, \( E \) only stores intersection points of line-segments adjacent in \( L \), and the intersection points are deleted from \( E \) whenever the intersecting segments become non adjacent.
The decision problem
The decision problem

**Input:** $n$ line-segments in the plane, $s_i = (p_i, q_i), \ i = 1 \ldots n$.

**Output:** there is / there is not a pair of intersecting line-segments, and report a witness.
The decision problem

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**Solution**

Bentley-Ottman’s algorithm solves this problem in \( O(n \log n) \) time.
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Lower bound

The decision problem has complexity $\Omega(n \log n)$. 
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Lower bound

The decision problem has complexity \( \Omega(n \log n) \).

*Proof:* by reduction from unicity of integers.

Given \( x_1, \ldots, x_n \in \mathbb{N} \), compute \( p_i = (x_i, 0), q_i = (x_i, 1) \) and \( s_i = (p_i, q_i) \).

There exists a pair of intersecting line-segments if and only if there exist duplicate numbers in the original set.

If you don’t like degeneracies, consider the following points:

\( p_i = (x_i - \frac{1}{2i}, 0) \) and \( q_i = (x_i + \frac{1}{2i}, 1) \).
The problem of reporting all intersections
The problem of reporting all intersections

Corollary
The problem of reporting all intersection has complexity $\Omega(k + n \log n)$, because

- Reporting requires $\Omega(k)$ time
- Deciding requires $\Omega(n \log n)$ time
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Optimal algorithm
An algorithm by Chazelle and Edelsbrunner solves this problem in $\Theta(k + n \log n)$ time.
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Consequences
- Deciding whether a polygon is simple can be solved in $O(n \log n)$ time.
- Deciding whether two simple polygons intersect can be solved in $O(n \log n)$ time.
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