PROXIMITY

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Let $P$ be a set of $n$ points in $\mathbb{R}^2$...
CLOSEST PAIR
Find the pair of points closest to each other
CLOSEST PAIR
Find the pair of points closest to each other

- Brute force solution: check every pair in $O(n^2)$ time
- This procedure does not exploit the metric properties
- In dimension 1, it can be solved in $O(n \log n)$ time by sorting the input first and using the fact that the solution must be a pair of consecutive points

APPLICATION
Airplanes in danger of collision
PROXIMITY

**All NEAREST NEIGHBORS**

Build the directed graph where the existence of an (oriented) edge $\overrightarrow{p_i p_j}$ means that $p_j$ is the point of $P$ closest to $p_i$.
PROXIMITY

**All NEAREST NEIGHBORS**

Build the directed graph where the existence of an (oriented) edge $\overrightarrow{p_ip_j}$ means that $p_j$ is the point of $P$ closest to $p_i$.

- Brute force solution: check every pair in $O(n^2)$ time
- This process does not exploit the metric properties
- In dimension 2, at most 6 points of $P$ have $p_i$ as their closest point
- In dimension 1 it can be solved in $O(n \log n)$ time (same as before)

**APPLICATION**

In Ecology, to study the territoriality of species
EUCLIDEAN MINIMUM SPANNING TREE

Build the tree connecting all points of $P$ and minimizing the sum of the lengths of its edges.
EUCLIDEAN MINIMUM SPANNING TREE

Build the tree connecting all points of $P$ and minimizing the sum of the lengths of its edges

- Applying Prim’s or Kruskal’s algorithms to the complete graph of $P$, with euclidean weights: $O(e \log e) = O(n^2 \log n)$ time
- Applying certain algorithms to the complete weighted graph: $O(e) = O(n^2)$ time
- These procedures do not exploit the metric properties
- In dimension 1 it can be done in $O(n \log n)$ time (same as before)

APPLICATION

Connections in general, telephone prices in the US
MIN-MAX FACILITY LOCATION: MINIMUM SPANNING CIRCLE

Find the point \( x \) on the plane achieving

\[
\min_{x \in \mathbb{R}^2} \max_{p_i \in P} d(x, p_i)
\]

Geometrically: find the center of the circle of minimum radius enclosing \( P \)
MIN-MAX FACILITY LOCATION: MINIMUM SPANNING CIRCLE

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$$\min_{x \in \mathbb{R}^2} \max_{p_i \in P} d(x, p_i)$$

Geometrically: find the center of the circle of minimum radius enclosing $P$

- Brute force solution: consider every set of three points, in $O(n^4)$ time
- In dimension 1 it can be solved in $O(n)$ time: the solution is the mid-point of $\min(P)$ and $\max(P)$

APPLICATION

Location of emergency services (hospitals, fire departments, ...), of radio and tv repeaters, antennas for mobile phones,...
MAX-MIN FACILITY LOCATION: MAXIMUM EMPTY CIRCLE

Find the point $x$ in the region $A$ achieving

$$\max_{x \in A} \min_{p_i \in P} d(x, p_i)$$

Geometrically: find in $A$ the center of circle of maximum radius not enclosing any point of $P$
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- Brute force solution: $O(n^4)$ time (as before)
- In dimension 1 it can be solved in $O(n \log n)$ time by sorting the input first and using the fact the solution is the mid-point of the maximum gap of $P$

APPLICATION
Location of undesired services (polluting industry, landfills, ...) or new shopping malls in competence with preexistent centers
NEAREST NEIGHBOR QUERY

Given a new point $q$, quickly find its closest site $p_i$. 

Discrete and Algorithmic Geometry, Facultat de Matemàtiques i Estadística, UPC
PROXIMITY

NEAREST NEIGHBOR QUERY
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NEAREST NEIGHBOR QUERY

Given a new point $q$, quickly find its closest site $p_i$

- Brute force: compare $q$ with every site $p_i$, in $O(n)$ time
- Preprocessing $P$ in order to speed up the queries is worth the effort when the query is to be repeatedly called
- In dimension 1, first sort the points of $P$ in $O(n \log n)$ time and then each query will be answered in $O(\log n)$ time by binary search

APPLICATION

The post office problem, pattern (voice) recognition and, in general, all classifications
• Structure capturing the proximity information

• Decomposes the plane into regions, each one is associated by proximity to one of the points of $P$

• Given two points, the boundary separating the portion of the plane closer to one point than to the other is the perpendicular bisector of the segment connecting the two points
• Structure capturing the proximity information

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History of the rediscoveries of the Voronoi diagram

- **Descartes**: distribution of matter in the solar system and surroundings (1644).


History of the rediscoveries of the Voronoi diagram

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- **Crystallography**: regular distribution of the points *Wirkung Sbereich, action domain, activity area, influence area* (end of the XIX century).

- **Meteorology**: rainfall average estimation based on local data, *Thiessen’s polygons* (1911).

- **Geology**: estimation of the presence of gold in a sediment, based on trial pits, *influence area polygons* (1909).


- **Ecology**: survival of organisms in competition for nourishment or light, like trees in the woods, *area potentially available* (1965), *plants polygons* (1966)

- **Medicine**: region of muscle tissue supplied by a capillary, *capillarity domains* (1985)
History of the rediscoveries of the Voronoi diagram (continued)

- **Economics:** market areas of centers in competition, under various conditions of market prices and transportation costs (since the mid XIX century)

- **Archaeology:** study of the diffusion of the use of tools in order to analyze the influence of the civilization centers (sixties-seventies)

- **Anthropology:** modelling territorial human organization under several aspects (sixties-seventies)
Ireland counties. On the left, theoretical partition created with the Voronoi diagram of the counties capitals. On the right, the real counties.

The school districts at Tsukuba. Left, real districts and the location of the schools (circles). Right, relocation of the schools (squares) so to minimize the area of the students that do not attend their closest school.

More recent applications of the Voronoi diagram

Interpolation of discrete sample values

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More recent applications of the Voronoi diagram


Interpolation of discrete sample values
Constructing the Voronoi diagram

Sixties
- Application concepts clear
- Lack of tools to build the diagrams

Seventies
- Start of the development of algorithm to compute the diagrams, thanks to computer science
- Discovery of new properties associated to the diagrams
PROXIMITY

DEFINITION

Let \( P = \{p_1, \ldots, p_n\} \) be a finite set of points in the plane.

The **Voronoi diagram** of \( P \) is the decomposition of the plane \( V(P) = \{V(p_1), \ldots, V(p_n)\} \) such that \( V(p_i) = \{x \mid d(x, p_i) \leq d(x, p_j) \ \forall \ j \neq i\} \).
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CHARACTERIZATION
If \( b_{ij} \) is the perpendicular bisector of the segment \( p_ip_j \), and \( H_{ij} \) is the halfplane defined by \( b_{ij} \) enclosing \( p_i \), then
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PROXIMITY

PROPERTIES

1. \(V(p_i)\) is a convex polygonal region.

2. \(V(p_i)\) is bounded if and only if \(p_i\) lies in the interior of \(ch(P)\).

3. Voronoi edges can be:
   - lines, if the points of \(P\) are all aligned;
   - half-lines, if the two points determining the edge are consecutive vertices of \(ch(P)\);
   - segments, if at least one of the points lies in the interior of \(ch(P)\).
4. The planarity of the Voronoi diagram allows applying Euler’s formula to the graph adding a vertex at infinity:

\[ v + n = e + 1. \]

Since each vertex is incident to at least 3 edges, and each edge has exactly 2 endpoints,

\[ 2e \geq 3(v + 1), \]

from where we get:

\[ v \leq 2n - 5, \quad e \leq 3n - 6. \]

Thus, the complexity of \( V(P) \) is \( O(n) \).

5. Although each \( V(p_i) \) can have \( n - 1 \) edges, on average they have 6.
Proximity between points: two points of $P$ are neighbors if they share an edge in the Voronoi diagram of $P$.

A proximity graph is obtained by connecting each point to its Voronoi neighbors. This graph is called Delaunay graph.

The Delaunay graph is the rectilinear dual of the Voronoi diagram.

In general, it is a triangulation of $P$, although sometimes it is not:

- when the points are aligned,
- when 4 or more points are concyclic.
THE DUAL GRAPH

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For any point \( q \) in the plane, let \( C_m(q) \) be the largest circle, centered at \( q \), which does not contain any point of \( P \) in its interior.

**Theorem**

A point \( q \) is a Voronoi vertex iff \( C_m(q) \) has three or more points of \( P \) in its boundary.
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A point \( q \) is a Voronoi vertex iff \( C_m(q) \) has three or more points of \( P \) in its boundary.

The bisector \( b_{ij} \) of two points \( p_i \) and \( p_j \) defines a Voronoi edge iff there exists a point \( q \) of the plane such that \( C_m(q) \) has \( p_i \) and \( p_j \) and no other point of \( P \) in its boundary. In such case, the Voronoi edge is the set of all these points \( q \).
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Corollary (blue and red)

Given any partition of $P$ into two classes (blue points and red points), the shortest red-blue segment is a Delaunay edge, because the circle having it as diameter is empty.
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Application 1
Closest neighbors digraph

For each point $p_i \in P$, the closest point of $P$ is a Voronoi neighbor. Therefore, the closest neighbors digraph is a subgraph of the Delaunay graph.
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Application 2

The closest pair

In particular, the shortest edge $p_ip_j$ belongs to the Delaunay graph.
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Application 3
Euclidean minimum spanning tree

The euclidean minimum spanning tree is a subgraph of the Delaunay graph.
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The euclidean minimum spanning tree is a subgraph of the Delaunay graph.

Prim’s algorithm to compute the minimum spanning tree in a weighted graph builds the tree \( T \) by adding, at each iteration, the edge of minimum weight incident to \( T \) which does not close a cycle when added to \( T \).
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At each step, if we consider the vertices of $T$ to be red, and the remaining ones to be blue, the added edge belongs to the Delaunay graph.
Application 4
Finding the closest site

Given any point \( q \) in the plane, finding its closest site \( p_i \in P \) means detecting the Voronoi region \( V(p_i) \) containing \( q \).
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Given any point $q$ in the plane, finding its closest site $p_i \in P$ means detecting the Voronoi region $V(p_i)$ containing $q$. 
Application
Max-min facility location

The center of the larger empty circle, restricted to a given region $A$, can be located at:

- a Voronoi vertex,
- the intersection of a Voronoi edge with the boundary of $A$.
- a vertex of $A$. 
Application
Max-min facility location

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Application 6
Min-max facility location

The center of the minimum spanning circle
Application 6
Min-max facility location

The center of the minimum spanning circle does not seem to be related with the Voronoi diagram! How strange!
GENERALIZATIONS OF THE VORONOI DIAGRAM

The Voronoi diagram of a finite set $P$ of points in the plane is a partition of the plane such that each region is the locus of the points that lie closer to one element of $P$ than to the remaining ones.
GENERALIZATIONS OF THE VORONOI DIAGRAM

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- segments
- disks
- barriers
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APPLICATIONS
- How do car emissions affect the trees along the highways.
- Rivers and mountains influence in determining school districts.
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sphere
cylinder
dimension \( d \)
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APPLICATIONS
- Sphere: study of global Earth problems
- Cylindre: study of problems with cyclic periodicity in time
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other metrics
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**APPLICATION**
Massive storage system with a reading/writing head based on the Manhattan or the maximum metrics.
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Numbers indicate multiplicative weights
GENERALIZATIONS OF THE VORONOI DIAGRAM

The Voronoi diagram of a finite set $P$ of points in the plane is a partition of the plane such that each region is the locus of the points that lie closer to one element of $P$ than to the remaining ones.

other metrics

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APPLICATIONS
- Population of each city
- Amount of emissions of some polluting product
- Size of each atom in a crystalline structure
- Combination of transportation prices and costs

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**APPLICATIONS**
- Finding the $k$-th closest emergency service
- Spatial interpolation in statistics estimations
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The Voronoi diagram of order $k = n - 1$ is the *Furthest Point Voronoi diagram*. 
GENERALIZATIONS OF THE VORONOI DIAGRAM

The Voronoi diagram of a finite set \( P \) of points in the plane is a partition of the plane such that each region is the locus of the points that lie closer to one element of \( P \) than to the remaining ones.

The Voronoi diagram of order \( k = n - 1 \) is the *Furthest Point Voronoi diagram*. It solves the minimum spanning circle problem!
Consider the point set \( P \) as being embedded in the plane \( z = 0 \).

Consider the paraboloid \( z = x^2 + y^2 \).

For each point \( p_i \in P \) let \( p_i^* \) be its vertical projection of \( p_i \) onto the paraboloid, i.e.:

\[
\text{if } p_i = (a_i, b_i, 0), \text{ then } p_i^* = (a_i, b_i, a_i^2 + b_i^2).
\]

For each point \( p_i^* \) consider the plane which is tangent to the paraboloid at \( p_i^* \).
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For each point $p_i \in P$ let $p_i^*$ be its vertical projection of $p_i$ onto the paraboloid, i.e.:

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The (farthest point) Voronoi diagram of $P$ is the orthogonal projection onto the plane $z = 0$ of the polyhedral convex region obtained when intersecting the upper (lower) halfspaces defined by these planes.
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THREE BIG QUESTIONS

1. How to build the Voronoi diagram of \( P \)?

2. How to store the Voronoi diagram of \( P \)?

3. How to use the Voronoi diagram to improve the solutions of the proximity problems?
TWO ADDRESSES TO PLAY
WITH VORONOI DIAGRAMS

http://www.pi6.fernuni-hagen.de/GeomLab/VoroGlide/

http://www.dma.fi.upm.es/docencia/segundociclo/geomcomp/voronoi.html

AND TWO BOOKS WITH MUCH MORE INFORMATION

A. Okabe, B. Boots, K. Sugihara, S. N. Chiu
*Spatial Tessellations*

F. Aurenhammer, R. Klein, D.-T. Lee
*Voronoi Diagrams and Delaunay Triangulations*