POINT LOCATION IN PLANAR SUBDIVISIONS

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Introduction
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Given a planar subdivision defined by a planar and rectilinear graph of size $n$, decide in which region of the decomposition is located a given point $p$. 
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Strategy

Adequately preprocess the planar decomposition, so that point locations can be efficiently performed.
Triangulation refinement: Kirkpatrick’s algorithm
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A triangulation $T$, whose exterior face is a triangle.
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A triangulation $T$, whose exterior face is a triangle.

**Preprocessing**
Create a hierarchy of triangulations $S_0, S_1, \ldots, S_h$ such that:

- $S_0 = T$
- $S_i$ is obtained from $S_{i-1}$ as follows:
  1. Delete a set of independent vertices not belonging to the boundary of the convex hull, as well as all their incident edges.
  2. Retriangulate the resulting polygons.
- $S_h$ is the enclosing triangle
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No two vertices are adjacent.
Preprocessing

\[ S_0 \]
Preprocessing
Preprocessing
Preprocessing
Preprocessing
Preprocessing

$S_1$
Preprocessing
Preprocessing
Preprocessing

$S_2$
Preprocessing

$S_2$
Preprocessing

\( S_3 \)
Preprocessing

The search structure to be build is a directed tree:

- The vertices of the tree are the triangles of the hierarchy of triangulations.
- There exists an edge from triangle $T_k$ to triangle $T_j$ if, when computing $S_i$ from $S_{i-1}$:
  - $T_j$ is deleted from $S_{i-1}$ in step 1.
  - $T_k$ is created in $S_i$ in step 2.
  - $T_k \cap T_j \neq \emptyset$. 
POINT LOCATION: Triangulation refinement

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If, at each step, the number of vertices in the independent set is a constant fraction $cn$ of the current number of vertices ($0 < c < 1$), then $h = O(\log n)$. 
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**Proof:**
The final number of vertices is $3 = n(1 - c)^h$, therefore $h = \frac{\log n - \log 3}{\log (1 - c)} = O(\log n)$. 

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Choosing in a greedy way all possible vertices of degree $\leq 8$ (as long as they stay independent), allows eliminating at each step at least $n/18$ vertices.
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Choosing in a *greedy* way all possible vertices of degree $\leq 8$ (as long as they stay independent), allows eliminating at each step at least $n/18$ vertices.

Order does not matter: choose one, label all its neighbors as being not independent, choose a second one, ...
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Choosing in a greedy way all possible vertices of degree $\leq 8$ (as long as they stay independent), allows eliminating at each step at least $n/18$ vertices.

**Proof:**
1. There exist at least $n/2$ vertices of order $\leq 8$. Otherwise, if more than half of the vertices had degree $\geq 9$, then $\sum$ degrees $\geq 9 \frac{n}{2} + 3(\frac{n}{2} - h) + 2h = 6n - 3$, and this cannot happen since $\sum$ degrees $= 2e = 2(3n - h - 3) = 6n - 12$.

2. Each time a vertex of degree $\leq 8$ is chosen, all its neighbors (at most 8 vertices) must be discarded. In the worst case, all of them will also be of degree $\leq 8$, and the process will be choosing 1/9 of the vertices of degree $\leq 8$. As there are $n/2$ such vertices, the minimum number of independent vertices is $n/18$. 
Complexity

**Space:** the space used to store the hierarchy of triangulations is $O(n)$:

- The total number of triangles is $O(n) + O((1 - c)n) + O((1 - c)^2 n) + \cdots = O(n)$.
- The number of pointers leaving a triangle is less or equal to the number of triangles that appear when retriangulating a hole of constant size $\leq 8$.

**Preprocessing:** computing the refinement of triangulations is done in $O(n)$ time, since at each step the algorithm:

- Finds the independent vertices (exploring all current vertices).
- Retriangulates a linear amount of holes, each of size $O(1)$.

Hence, the overall task is done in time $O(n) + O((1 - c)n) + O((1 - c)^2 n) + \cdots = O(n)$.

**Location:** locating a point is done in $O(\log n)$ time, since the height of the search tree is $h = O(\log n)$. 
Extension to arbitrary graphs

When the planar decomposition is not a triangulation but an arbitrary graph:

1. Enclose the graph in a triangle.
2. Triangulate all the resulting non triangular regions.
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This adds $O(n \log n)$ running time to the preprocessing step (except if the polygons are triangulated with Chazelle’s linear algorithm).
Extension to Voronoi diagrams

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Preprocessing

1. Consider a triangle enclosing all Voronoi vertices.
2. Clip each half-line with the boundary of the enclosing triangle.
3. Triangulate all the regions.

This can be done in $O(n)$ time.

Search

When a point lies in the exterior of the enclosing triangle, the algorithm must detect its relative position with respect to the half-lines. This can be done in $O(\log n)$ time (with the appropriate data structure), since the half-lines are sorted.
Walking in a triangulation
POINT LOCATION: Walking in a triangulation

This method is used for locating points in triangulations.
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If finds out in which triangle a new point $p$ lies, starting from a known vertex $q$ of the triangulation.
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**Rectilinear walk**

The algorithm visits all the triangles intersected by the line-segment \( pq \).
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**Correction**

- Rectilinear walk: obvious.
- Orthogonal walk: obvious.
- Visibility walk: works fine for some triangulations, but not for arbitrary triangulations.
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**Advantages and disadvantages**

- On average, rectilinear walks explore a smaller number of triangles.
- Orthogonal walks have the advantage that almost each test is in dimension 1. This is specially interesting when working in higher dimension.
- Visibility walks are easier to implement, because no degenerate positions need to be taken into account, as opposed to rectilinear walks, which require solving the case of the line-segment $pq$ containing a vertex of the triangulation.
- Deterministic visibility walks cannot be applied to arbitrary triangulations.
- All three walks can be generalized to higher dimension.
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**Number of intersected triangles**

For Delaunay triangulations on uniformly distributed random points, the expected number of visited triangles for each walk is:

- Rectilinear walk: \( O(|p - q|\sqrt{n}) \).
- Orthogonal walk: \( O(|p| + |q|\sqrt{n}) \).
- Visibility walk: there exist triangulations for which the expected number is \( > 2^{3\sqrt{n}} \).

In practice, in addition to the number of intersected triangles, the cost of each operation must be taken into account, as well as the effort of programming degenerated cases. Finally, the choice of point \( q \) can speed up or slow down the process.
Slab decomposition
Preprocessing

Decompose the plane into the slabs determined by all vertical lines through the vertices of the original decomposition.
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**Location**

Given a point $q$:

1. Locate the abscissa of $q$ in the corresponding slab.
2. Within the slab, locate the two segments between which $q$ lies.
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The preprocessing should:

- Store the abscissae of the slabs in a structure allowing binary searching.
- Store the segments intersecting each slab in a structure allowing binary searching.
- For each segment of the initial decomposition, store a pointer to the face above (or below) it.

In this way, it will be possible to perform each point location in $O(\log n)$ time.
**Preprocessing**

It can be done by sweeping the planar decomposition with a vertical line:

**Event queue:** the vertices of the decomposition, sorted by their abscissae.

**Sweep line status:** the line-segments of the decomposition stabbed by the line, in order.

**Action at each event:**

1. In the sweep line:
   - Insert vertex: insert the incident edges in the sweep line, in counterclockwise order.
   - Delete vertex: delete the incident edges from the sweep line.
   - Update vertex: delete the edges incident to the left and insert the edges incident to the right, in sorted order.

2. In the slab decomposition:
   - Store the line-segments of the new slab in order.
Complexity
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Some planar decompositions have a quadratic slab decomposition:

In this example, if the total number of vertices is $2n + 1$, the total number of regions of the slab decomposition is

$$3 + (2n + 1) + 2 \sum_{i=1}^{n-1} (2i + 1) \geq 2 \sum_{i=1}^{n-1} 2i =$$

$$= 4 \sum_{i=1}^{n-1} i = 4 \frac{n(n - 1)}{2} = O(n^2).$$
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**Space:** the space needed to store the information is $O(n^2)$.

**Preprocessing:** the preprocessing is done in $O(n^2)$ time.

**Location:** locating of a point is done in $O(\log n)$ time.
Trapezoidal refinement
This method is a variation of the slab decomposition.

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**Trapezoid characteristics**

The trapezoids used in this method have the following characteristics:

- Their vertical edges are line-segments or half-lines through vertices of the initial decomposition.
- The other edges of the trapezoid are edges or portions of edges of the initial graph.
- No edge of the initial graph simultaneously intersects both vertical edges of a trapezoid.
Refinement

Given a trapezoid $T$: 
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1. Consider the vertical line through the vertex of median abscissa among all vertices in $T$. 
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3. If $T_1$ and $T_2$ are legal trapezoids, no further action is needed.
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1. Consider the vertical line through the vertex of median abscissa among all vertices in $T$.

2. Decompose $T$ into two strips, $T_1$ and $T_2$, left and right of the line.

3. If $T_1$ and $T_2$ are legal trapezoids, no further action is needed.

4. For each edge completely traversing $T_1$ (respectively $T_2$) -recall that no edge can traverse both-, decompose $T_1$ ($T_2$) into two pieces, one above and one below the edge.
Search structure
Search structure

POINT LOCATION: Trapezoidal refinement
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Search structure

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Complexity

Space: the space used to store the hierarchy of trapezoidal decompositions is $O(n \log n)$:

- There is a triangular node for each vertex of the initial graph.
- There is a circular node for each “piece” of an edge of the initial graph.
- There is a leaf for each trapezoid free of vertices in its interior.

It can be proved that this hierarchy cannot produce more than $O(n \log n)$ overall trapezoids.

Preprocessing: computing the refinement of the trapezoids is done in $O(n \log n)$ time.

Location: locating a point is done in $O(\log n)$ time.
Trapezoidal map
This is a randomized method, very convenient in practice.

It consists in a variation of the slab method:

1. Compute a rectangle enclosing the 1-skeleton of the graph $G$.

2. From each vertex of $G$, shoot two vertical rays, upwards and downwards, until they reach an edge (of the graph or of the enclosing rectangle).
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The total number of resulting trapezoids is linear: for each vertex of $G$, at most two new edges and two new vertices are added.
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If the line-segments of the initial decomposition $G$ are processed in random order:

**Space:** The appropriate search structure has expected size $O(n)$.

**Preprocessing:** The expected running time for computing the trapezoids and building the search structure is $O(n \log n)$.

**Location:** Given a point $q$ of the plane, the region where it is located is found in $O(\log n)$ expected running time.
Monotone subdivision
This method is based on the following observation: if $C$ is an $x$-monotone chain with $n$ vertices, deciding whether a point $q$ lies above or below $C$ can be done in $O(\log n)$ time.

1. Locate the abscissa $q_x$ of $q$ between the abscissae $x_i$ and $x_{i+1}$ of two consecutive vertices of $C$, by binary search.

2. Decide whether $q$ lies above or below the line-segment $p_ip_{i+1}$ of $C$.
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Preprocessing

Compute \( C \), a complete set of \( x \)-monotone chains for the planar decomposition:

- The union of the chains of \( C \) contains the 1-skeleton of the decomposition.
- If \( C_i \) and \( C_j \) are two chains of \( C \), all the vertices of \( C_i \) which do not belong to \( C_j \) lie to the same side of \( C_j \). This implies that \( C \) is a totally ordered set.

Therefore, if \( C \) contains \( r \) chains, and the largest of the chains has \( k \) vertices, it is possible to locate a point between two consecutive chains in \( O(\log r \log k) \) time by binary search.
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Compute $\mathcal{C}$, a complete set of $x$-monotone chains for the planar decomposition:

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Therefore, if $\mathcal{C}$ contains $r$ chains, and the largest of the chains has $k$ vertices, it is possible to locate a point between two consecutive chains in $O(\log r \log k)$ time by binary search.
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Therefore, if $C$ contains $r$ chains, and the largest of the chains has $k$ vertices, it is possible to locate a point between two consecutive chains in $O(\log r \log k)$ time by binary search.
Conditions

For a planar decomposition to admit a complete set of $x$-monotone chains, the graph $G$ is required to be regular:

- Consider the vertices $v_1, v_2, \ldots, v_n$ of $G$ to be lexicographically sorted:

  $i < j \iff x(v_i) < x(v_j) \text{ or } (x(v_i) = x(v_j) \text{ and } y(v_i) < y(v_j)).$

- A vertex $v_i$ is regular if $\exists j, k$ with $j < i < k$ such that $v_jv_i$ and $v_iv_k$ are edges of $G$.

- The graph $G$ is regular if all its vertices, other than $v_1$ and $v_n$, are regular.
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If $G$ is a regular graph, then it admits a complete set of $x$-monotone chains.
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Proof:

1. By induction, we will prove that it is possible to compute an $x$-monotone chain from $v_1$ to $v_i$ for all $i$:
   
   • For $i = 2$ the result is trivial, because $\exists j < 2$ such that $v_jv_2$ is an edge of $G$.
   • Assume the result true for all $j < i$. Due to the regularity of $v_i$, $\exists j < i$ such that $v_jv_i$ is an edge of $G$. By induction hypothesis, there exists an $x$-monotone chain $C$ from $v_1$ to $v_j$. The concatenation of $C$ and $x_jx_i$ is $x$-monotone.

2. In the following, we will prove that it is possible to build a complete set $C$ of $x$-monotone chains.
Theorem

Notation:

• $in(v_i) = \{\text{edges } v_jv_i \text{ of } G \text{ with } j < i\}$.

• $out(v_i) = \{\text{edges } v_iv_k \text{ of } G \text{ with } i < k\}$.

• $w(e) = \text{weight of edge } e = \text{number of chains of } C \text{ in which } e \text{ appears}.$

• $w_{in}(v) = \text{ingoing weight of vertex } v = \sum_{e \in in(v)} w(e)$.

• $w_{out}(v) = \text{outgoing weight of vertex } v = \sum_{e \in out(v)} w(e)$.
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It can be proved:

- \(\forall e \quad w(e) > 0.\)  
  This implies that the union of the chains of \(C\) contains the 1-skeleton of \(G\).

- \(\forall i \neq 1, n \quad w_{in}(v_i) = w_{out}(v_i).\)  
  This implies that \(C\) can be constructed so that it is a totally ordered set.
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Weight assignment:
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1. *Initialization*
   Assign weight 1 to all edges
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2. **First sweep (forwards)**
   From $i = 2$ to $i = n - 1$ do:
   If $w_{in}(v_i) > w_{out}(v_i)$, replace the weight 1 of the (counterclockwise) first outgoing edge of $v_i$ by the weight $w_{in}(v_i) - w_{out}(v_i) + 1$. 

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3. **Second sweep (backwards)**
   
   From $i = n - 1$ to $i = 2$ do:
   
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Computing the chains:

- All chains start at $v_1$ and end at $v_n$.
- Each time an edge $e$ is used to build a chain, its weight $w(e)$ is decreased by one.
- During the construction, always leave vertices through the (counterclockwise) first edge having positive weight.
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This seems to mean that information of size $O(n^2)$ should be stored, but this is redundant: the graph can be stored in $O(n)$ space, based on the following observation:
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This seems to mean that information of size $O(n^2)$ should be stored, but this is redundant: the graph can be stored in $O(n)$ space, based on the following observation:

When an edge $e$ belongs to several chains, it belongs to an interval of consecutive chains (chains are totally ordered). Store $e$ in the ascendant of the entire interval of chains in the search structure $C$. It is in that chain where $e$ will be used in the search algorithm. For each remaining chain of the interval, the edge $e$ is replaced by a bypass pointer.

The number of pointers is not quadratic, but linear, because each pointer points to one single edge of $G$, and each edge gets at most one single pointer.
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**Preprocessing:** If the graph is regular, the preprocessing can be performed in $O(n)$ time from the DCEL of $G$, assuming that the vertices of $G$ are sorted by abscissa. Otherwise, sorting the vertices is necessary as first step, requiring $O(n \log n)$ time.
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We have already observed that the search running time is $O(\log r \log k)$, where $r$ is the number of chains and $k$ is the maximum number of vertices of the chains. The following example shows $r, k \in O(\sqrt{n})$, so that $O(\log r \log k) = O(\log^2 \sqrt{n}) = O(\log^2 n)$:
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In fact, the running time of the location step can be decreased by taking into account the following observation: when testing a point $q$ wrt a chain $C$, the algorithm ends testing $q$ wrt some particular edge $e$. At the following step, this information can be used to avoid locating $q$ along the entire chain when, in fact, only a portion of the chain matters.
Extension to non-regular graphs

Every non-regular graph can be regularized in $O(n \log n)$ time.
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![Diagram of the process]

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$G$ can be regularized by sweeping it twice: once from left to right to regularize all vertices without ingoing edges, once from right to left to regularize all vertices without outgoing edges.

The resulting algorithm runs in $O(n \log n)$ time.
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Let $v \neq v_1$ be a vertex without any ingoing edge (the case of a vertex without outgoing edges is analogous).

Since $v \neq v_1$, the vertical line through $v$ intersects at least one, if not two, edges of $G$, $e_i$ and $e_j$, adjacent to $v$. Let $v_i$ and $v_j$ respectively be their left endpoints. Connect $v$ with the rightmost vertex among those lying in the trapezoid limited by $e_i$, $e_j$, and the vertical lines through $v$ and through the rightmost vertex among $v_i$ and $v_j$. The resulting line-segment does not intersect any edge of $G$; inserting it in $G$ regularizes vertex $v$.

$G$ can be regularized by sweeping it twice: once from left to right to regularize all vertices without ingoing edges, once from right to left to regularize all vertices without outgoing edges.

The resulting algorithm runs in $O(n \log n)$ time.
Point Location: Monotone subdivision

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