MINIMAL SET OF CONSTRAINTS FOR 2D CONSTRAINED DELAUNAY RECONSTRUCTION*

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ABSTRACT

Given a triangulation \( T \) of \( n \) points in the plane, we are interested in the minimal set of edges in \( T \) such that \( T \) can be reconstructed from this set (and the vertices of \( T \)) using constrained Delaunay triangulation. We show that this minimal set is precisely the set of non locally Delaunay edges, and that its cardinality is less than or equal to \( n + i/2 \) (if \( i \) is the number of interior points in \( T \)), which is a tight bound.

Keywords: Triangulation; Delaunay; 2D; reconstruction; minimal constraints set.

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1. Introduction

1.1. Motivations

In the very active field of geometric compression, the works that deal with meshes compression give generally a method to encode the whole topology of the geometric object.\textsuperscript{7,8,2} In some cases the topology can be computed from the geometry, for example some terrain models or some finite elements meshes are obtained by using the Delaunay triangulation; in these cases, alternative methods coding only the geometry can be used\textsuperscript{6,3} saving the cost of coding the topology. Unfortunately, not all triangulations are Delaunay triangulations, and coding a non Delaunay triangulation must include some topology, although in practice many edges behave locally like Delaunay edges. A method consists in coding only few constrained edges and then reconstruct the topology using the constrained Delaunay triangulation. This idea is exploited in particular by Kim et al. to achieve terrain models compression.\textsuperscript{4}

This leads to the two theoretical problems that are addressed in Section 2 and Section 3:

- given a 2-dimensional triangulation $T$, compute $E$ the minimal subset of the edges of $T$ such that $T$ is the constrained Delaunay triangulation of $E$,
- find the theoretical worst-case bound for this minimal set.

We give in Section 4 statistics on the number of non Delaunay edges in some geographic databases.

1.2. Basic definitions

Definition 1 (Delaunay criterion). Let $p_1 p_2$ be an edge in a 2-dimensional triangulation $T$. We say that $p_1 p_2$ is a Delaunay edge if there exists a circle going through $p_1$ and $p_2$ empty of points of $T$ (see Figure 1).

Definition 2 (Local Delaunay criterion). Let $p_1 p_2$ be an edge in $T$, and let \{ $p_1 p_2 p_3$ \} and \{ $p_1 p_2 p_4$ \} be the triangles adjacents to $p_1 p_2$. We will say that $p_1 p_2$ is a locally Delaunay edge if the circle ($p_1 p_2 p_3$) does not contain $p_4$ or equivalently if the circle ($p_1 p_2 p_4$) does not contain $p_3$ (see Figure 2).

Remark 1. In particular, if the quadrilateral \{ $p_1 p_2 p_3 p_4$ \} is non convex with reflex angle in $p_1$ or $p_2$, then $p_1 p_2$ is locally Delaunay (see Figure 3). Convex hull edges can also be considered as locally Delaunay.

Definition 3 (Edge Flip). Let $p_1 p_2$ be an edge in $T$, and let $p_3$ and $p_4$ be the vertices of its adjacent triangles. We say that $p_1 p_2$ is flipped when it is replaced by $p_3 p_4$ in $T$. This is possible only if \{ $p_1, p_2, p_3, p_4$ \} is a convex quadrilateral (see Figure 4).
Definition 4 (Constrained Delaunay triangulation). Given a set of points $P$ and a set of edges $E$ in the plane, the constrained Delaunay triangulation $CD(P, E)$ is the unique triangulation such that each of its edges is either in $E$ or locally Delaunay.

Remark 2. This definition is equivalent to the classical definition used for example by Chew¹: $CD(P, E)$ is the unique triangulation containing $E$ and such that for each remaining edge $e$ of $CD(P, E)$, there exists a circle $c$ with the following properties:
(1) The endpoints of edge $e$ are on the boundary of $c$,
(2) If any vertex $v$ of $E$ is in the interior of $c$ then it cannot be “seen” from at least one of the endpoints of $e$.

This equivalence is shown in particular in an article by Lee and Lin.  

2. Minimal Set of Constraints

**Theorem 1.** Let $T$ be a 2-dimensional triangulation, $P_T$ the set of its vertices, and $NLD_T$ the set of its edges that are not locally Delaunay. Then $NLD_T$ is the minimal set $S$ such that $CD(P_T, S) = T$.

**Proof.** It is easy to see that $CD(P_T, NLD_T) = T$. Indeed, $NLD_T \subset T$ so we can complete $NLD_T$ to obtain $T$. But doing that, we add $T \setminus NLD_T$, which, by definition, consists of the locally Delaunay edges of $T$ only. Therefore there is no Delaunay flippable edges and the constrained Delaunay triangulation is over. So $NLD_T$ is a sufficient set of constraints.

Reciprocally, let us show that $NLD_T$ is necessary. Let $W_T$ be a subset of the edges in $T$ such that $CD(P_T, W_T) = T$. Let $e$ be an edge in $NLD_T \setminus W_T$, and $\{e_1, e_2, e_3, e_4\}$ the edges and $\{p_1, p_2, p_3, p_4\}$ the vertices of the correspond-
ing quadrilateral (which is convex by remark 1). Since $CD(P_T, W_T) = T$, then $CD(P_T, W_T \cup \{e_1, e_2, e_3, e_4\}) = T$. This implies in particular that $e \in CD(\{p_1, p_2, p_3, p_4\}, \{e_1, e_2, e_3, e_4\})$, which is false since $e$ is non locally Delaunay. Hence such an edge $e$ cannot exist, and $NLD_T$ is minimal.

Remark 3. As a direct consequence, we obtain a linear algorithm to compute the minimal set of constraints of a 2-dimensional triangulation.

3. Worst Case Study

In this section, we will show that, given $n$ points triangulation in the plane, the maximal number of non locally Delaunay edges is half the total number of edges, and that this bound is tight.

Lemma 1. Given a triangulation $T$, any open half-plane $H$ whose boundary contains one interior point $p$ of $T$, contains at least one locally Delaunay edge incident to $p$.

Proof. Let $p_0, p_1, p_2 \ldots p_k$ be the vertices incident to $p$ in counterclockwise order such that $p_0 \not\in H$ and $p_1 \in H$ (since $p$ is interior there exist neighbors of $p$ both in $H$ and in its complementary); let $j$ be such that $p_1 \ldots p_j$ are the vertices belonging to $H$. If $p_1 = p_j$ (there is only one edge incident to $p$ in $H$) then $pp_1$ is necessarily locally Delaunay by Remark 1. Otherwise, lets assume for contradiction that all edges $pp_i$, $1 \leq i \leq j$ are non locally Delaunay. We denote the circle through $pp_i p_{i+1}$ by $C_i$ and by $R_{i+1}$ the region limited by the line segment $pp_{i+1}$ and the arc of circle $C_i$ from $p_{i+1}$ to $p$ counterclockwise (see Figure 5). Since $pp_1$ is non locally Delaunay, $C_1$ must contain the point $p_0$, and since $p_0$ is not in $H$ we get that $R_2 \subset H$. Now since $pp_2$ is non locally Delaunay, $C_1$ must contain the point $p_3$, more precisely $p_3 \in R_2 \subset H$ iterating the process we prove that for all $i$ $p_i \in R_{i-1} \subset H$ which contradicts $p_{j+1} \not\in H$, and thus at least one edge in $H$ incident to $p$ is locally Delaunay.

Theorem 2. Let $P$ be a set of $n$ points, $i$ of them being interior. Then every triangulation of $P$ contains at least $n + \left\lceil \frac{i+3}{2} \right\rceil$ locally Delaunay edges. Moreover, this bound is tight.

Proof. Lemma 1 clearly implies that for any interior point $p$ of $P$, there are at least 3 edges incident to $p$ in $T$ that are locally Delaunay. The $n - i$ exterior points (points of the convex hull of $P$) are incident to two convex hull edges that are locally Delaunay. Finally, if $p$ is a point of the convex hull of the $i$ interior point, one of the 3 edges incident to $p$ must link it to an exterior point, there is at least three such points and thus three other edges incident to exterior points. Thus we get at least $\left\lceil \frac{3i+2(n-i)+3}{2} \right\rceil = n + \left\lceil \frac{i+3}{2} \right\rceil$ locally Delaunay edges.
For tightness, since there are $2n - 3 + i$ edges in a triangulation, we just showed a triangulation containing exactly $n + \left\lceil \frac{i+3}{2} \right\rceil$ locally Delaunay edges.

The example is constructed as follows, first we recall the well known fact that for a set of points belonging to an arc of parabola with monotone variation of the curvature (the apex does not belong to the arc), the Delaunay triangulation links the point of highest curvature to all points and in the triangulation which links the point of smallest curvature to all others, all non convex hull edges are locally non Delaunay. We arrange three arcs of parabola as described on the left of Figure 6; the set of points is composed of 6 points described on the figure, of $n - i - 3$ points on arc $\alpha$, of $\left\lfloor \frac{i-3}{2} \right\rfloor$ on arc $\beta$, of $\left\lceil \frac{i-3}{2} \right\rceil$ on arc $\gamma$ and of one point on arc $\delta$ if $i$ is even (nothing if $i$ is odd). Then each parabola is triangulated with respect to its point of smallest curvature ($a$ for $\alpha$, $b$ for $\beta$ and $\gamma$) and in the central part, there are $\left\lfloor \frac{i-3}{2} \right\rfloor$ convex quadrilaterals in which we choose the non locally Delaunay diagonal. We get the triangulation of the right of Figure 6 which realize the minimum number of locally Delaunay edges. \qed
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<th>cocirc. edges (%)</th>
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Fig. 7. Statistics on the minimal constraints set of practical models.

Fig. 8. The "crater" terrain model.

4. Experimental Results

We propose here some statistical results on the size of the set $NLD_T$ in practice. The first example of the table comes from the viewpoint collection (http://avalon.viewpoint.com/), whereas the other terrain models tested can be found on a web site of the U.S. Environmental Protection Agency offering several triangulated irregular networks (TIN) in VRML format (http://www.epa.gov/gisvis/vrml/).

Due to the small number of bits used to store the points coordinates, degenerate cases of four cocircular points appears to be frequent. Referring to Definition 2, we call an edge $p_1p_2$ cocircular if the four points $p_1p_2p_3$ and $p_4$ are cocircular. We count in Figure 7 the percentage of non Delaunay edges, and the percentage of cocircular edges. The last two columns of the table show the compression ratios (in bits per vertex) obtained by the algorithm of Devillers and Gandoin. The topologic ratios include the coding of cocircular edges, achieved by an additional sequence of
1 bit per cocircular edge. The geometric ratios correspond to a quantization of 12 bits per coordinate.

5. Conclusion

In the information needed to represent a triangulation, we distinguish a geometric part (the vertex positions) and a topological part (the edges). We prove in Theorem 1 that if only the non locally Delaunay edges of a triangulation are stored, then other edges can be reconstructed by constrained Delaunay triangulation. We also prove that this number cannot exceed half of the total number of edges (Theorem 2).

Although this bound is tight on some pathological examples, our experiments on real data sets shows a practical rate of non locally Delaunay edges of less than 3%, which yields to a very effective compression of the topological part of the triangulation.

For three-dimensional meshes, the definition of the constrained Delaunay triangulation (definition 4) generalizes straightforwardly. Unlike the 2D case, given a set of constraints, the CDT does not always exist. However, given a triangulation \( T(P, E) \) and its non locally Delaunay faces \( NLD_T, CD(P, NLD_T) \) exists and Theorem 1 generalizes easily.

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