Small Weak Epsilon-Nets

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Abstract

Given a set P of points in the plane, a set of points Q is a *weak ε-net* with respect to a family of sets $\mathcal{S}$ (e.g., rectangles, disks, or convex sets) if every set of $\mathcal{S}$ containing $\varepsilon|P|$ points contains a point of Q. In this paper, we determine bounds on $\varepsilon_i^{\mathcal{S}}$, the

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smallest epsilon that can be guaranteed for any P when $|Q| = i$, for small values of i.

Key words: weak epsilon-nets, rectangles, set systems, convex sets.

1 Introduction

Let P be a set of n points in $\mathbb{R}^2$. A point q (not necessarily in P) is called a *centerpoint* of P if each closed half-plane containing q contains at least $\lceil \frac{n}{3} \rceil$ points of P, or, equivalently, any convex set that contains more than $\frac{2}{3}n$ points of P must also contain q. It is a well known fact that a centerpoint always exists and the constant $\frac{2}{3}$ is the best possible (see, e.g., [13] for more details). Can we improve this constant by using, say, two points, or some other small number of points? What happens when we replace convex sets by, say, axis-parallel rectangles? In this paper we address such questions. We start by generalizing the notion of a centerpoint.

Definition 1 [9] *Let P be an n-point set in $\mathbb{R}^2$. Consider a family $\mathcal{S}$ of sets in $\mathbb{R}^2$. A set $Q \subset \mathbb{R}^2$ is called a* weak ε-net *for P with respect to $\mathcal{S}$, if for any $S \in \mathcal{S}$ with $|S \cap P| > \varepsilon n$, we have $S \cap Q \neq \emptyset$. Further, if $Q \subseteq P$, then Q is called a* (strong) ε-net *for P with respect to $\mathcal{S}$.*

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Any centerpoint of P constitutes a 1-point weak $\frac{2}{3}$-net of P with respect to the class of convex sets. The concepts of ε-net and weak ε-net were first defined by Haussler and Welzl [9] and quickly found many applications in range searching, approximation algorithms, and geometric optimization. When the VC-dimension [7] of the range space $(\mathbb{R}^2, \mathcal{S})$ is some constant d, an ε-net (and therefore also a weak ε-net) of size $(d/\varepsilon)\ln(1/\varepsilon) + O((d/\varepsilon)\ln\ln(1/\varepsilon))$ always exists [9,11], for any P (and n). When $\mathcal{S}$ is the family of all convex sets, the VC-dimension of the range space $(\mathbb{R}^2, \mathcal{S})$ is infinite and the previous bound does not apply. Nevertheless, it is known that, for any $\varepsilon > 0$ and for any set P of points in the plane, there exists a *weak* ε-net for P with respect to the set $\mathcal{S}$ of all convex sets of size at most $O(\frac{1}{\varepsilon^2})$; see [1]. The best known lower bound is the trivial $\Omega(\frac{1}{\varepsilon})$ bound, which holds already when all points are on a line. Recent works analyze the size of weak ε-nets for specific classes of point sets. In [2] the authors construct weak ε-nets of almost linear size for certain types of point sets: (1) for planar point sets in convex position they construct weak $\frac{1}{r}$-nets of size $O(r\alpha(r))$, where $\alpha(r)$ denotes the inverse Ackermann function; (2) for point sets on the moment curve in $\mathbb{R}^d$ they construct weak $\frac{1}{r}$-nets of size $r \cdot 2^{poly(\alpha(r))}$, where the degree of the polynomial in the exponent depends (quadratically) on d. See, e.g., [13] for more details on weak ε-nets.

In this paper, we consider weak ε-nets of small constant size. Let $0 \le \varepsilon_i^{\mathcal{S}} \le 1$ denote the smallest real number such that, for any finite point set $P \subset \mathbb{R}^2$, there exists a weak $\varepsilon_i^{\mathcal{S}}$-net for P with respect to $\mathcal{S}$ of size i. We provide upper and lower bounds for $\varepsilon_i^{\mathcal{S}}$ for small sizes i, when $\mathcal{S}$ is the family of all convex sets, of all half planes, of all disks, or of all axis-parallel rectangles. Note that for range spaces of finite VC-dimension the ε-net theorem implies that $\varepsilon_i^{\mathcal{S}} = O(\frac{\log i}{i})$. However, it may very well be that in some cases such as geometrically induced range spaces the truth is $\varepsilon_i^{\mathcal{S}} = O(\frac{1}{i})$. For instance, this is open for the set of all axis-parallel rectangles. It therefore makes sense to resort to weak ε-nets, fix the size of the net, and try to see what is the best $\varepsilon_i^{\mathcal{S}}$ one can achieve with a net of size i.

Table 1 summarizes some of the bounds obtained in this paper. In all four set families considered, $\varepsilon_0^{\mathcal{S}} = 1$ since for all finite $P \subset \mathbb{R}^2$ there exists a set in the family that contains P. Bounds on $\varepsilon_i^{\mathcal{H}}$ where $\mathcal{H}$ is the family of all halfplanes are proved at the end of Section 2. Let $\mathcal{C}$ be the family of all convex sets in the plane, $\mathcal{D}$ the family of all disks in the plane, and $\mathcal{R}$ the family of all axis-parallel rectangles in the plane. Lower bounds on $\varepsilon_i^{\mathcal{C}}$ for $i \ge 4$ and on $\varepsilon_3^{\mathcal{D}}$ follow from general bounds proved in Section 2. The value of $\varepsilon_1^{\mathcal{C}}$ follows from the centerpoint theorem, all other bounds on $\varepsilon_i^{\mathcal{C}}$ are proved in Section 3. The upper bound on $e_4^{\mathcal{D}}$ is shown in Section 5. Other bounds on $e_i^{\mathcal{D}}$ follow from the fact that $\mathcal{H} \subset \mathcal{D} \subset \mathcal{C}$, which implies $\varepsilon_i^{\mathcal{H}} \le \varepsilon_i^{\mathcal{D}} \le \varepsilon_i^{\mathcal{C}}$ by Lemma 2.2 below.

[7] The VC-dimension is an indicator of the combinatorial complexity of geometric set systems. See, e.g., [13].

	convex sets		half-planes		disks		rectangles	
	LB	UB	LB	UB	LB	UB	LB	UB
ε_0	1		1		1		1	
ε_1	2/3		2/3		2/3		1/2	
ε_2	5/9	5/8	1/2		1/2	5/8	2/5	
ε_3	5/12	7/12	0		1/4	7/12	1/3	
ε_4	1/5	4/7	0		1/5	1/2	1/5	5/16
ε_5	1/6	1/2	0				1/6	1/4

Table 1
Table of results.

Finally, bounds on $\varepsilon_i^{\mathcal{R}}$ are shown in Section 4.

2 General bounds

We first prove some very weak general bounds that will be used in subsequent sections.

Lemma 2.1 *If there exists a line L in the plane with the property that for every line segment on L there is a set $s \in \mathcal{S}$ such that $s \cap L$ is that segment, then $\varepsilon_i^{\mathcal{S}} \geq \frac{1}{i+1}$.*

Proof: Place $n = k(i+1)$ points on L, and divide them in $i+1$ disjoint groups each consisting of k consecutive points. If $\varepsilon_i^{\mathcal{S}} < \frac{1}{i+1}$, then each group has to contain one point from the net, so $i+1$ points are needed, a contradiction. □

The next lemma follows directly from the definition of weak ε-nets.

Lemma 2.2 *If $\mathcal{S} \subseteq \mathcal{S}'$ then $\varepsilon_i^{\mathcal{S}} \leq \varepsilon_i^{\mathcal{S}'}$ for all i.*

Let $\mathcal{H}$ denote the family of all half-planes.

Lemma 2.3 *$\varepsilon_1^{\mathcal{H}} = \frac{2}{3}$, $\varepsilon_2^{\mathcal{H}} = \frac{1}{2}$, and $\varepsilon_i^{\mathcal{H}} = 0$ for $i \geq 3$.*

Proof: The centerpoint theorem [13] proves that $\varepsilon_1^{\mathcal{H}} = \frac{2}{3}$.

For any point set P, let ℓ be a vertical line that bisects P, and pick two points q_1 and q_2 on ℓ above and below the convex hull of P. Thus any half-plane not containing q_1 or q_2 can only contain points of P from one of the two half-planes delimited by ℓ. This proves $\varepsilon_2^{\mathcal{H}} \leq \frac{1}{2}$. On the other hand, given a set of n points in general position, and any net $Q = \{q_1, q_2\}$ of size 2, one of the two open half-planes delimited by the line through q_1 and q_2 contains at least $\lfloor (n-2)/2 \rfloor \geq n/2 - 2$ points. Thus $\varepsilon_2^{\mathcal{H}} \geq 1/2 - 2/n$ for all n. As n can be chosen to be arbitrarily large, this proves $\varepsilon_2^{\mathcal{H}} \geq \frac{1}{2}$.

Given any point set P, pick $Q = \{q_1, q_2, q_3\}$ so that the triangle formed by those three points contains P. Thus any half-plane containing any point from P must contain at least one point of Q. This proves $\varepsilon_i^{\mathcal{H}} = 0$ for $i \geq 3$. $\square$

3 Convex sets

Let $\mathcal{C}$ denote the family of all convex sets in the plane. In this section, we derive various bounds on the quantity $\varepsilon_i^{\mathcal{C}}$, for $i \geq 2$. We start by proving a lower bound on $\varepsilon_2^{\mathcal{C}}$ and $\varepsilon_3^{\mathcal{C}}$.

Theorem 3.1 $\varepsilon_2^{\mathcal{C}} \geq \frac{5}{9}$ *and* $\varepsilon_3^{\mathcal{C}} \geq \frac{5}{12}$.

Proof: For any n, we construct a set P of n points such that, for any pair of points q, r in the plane, there is a convex set K that avoids q, r and contains at least $\frac{5}{9}n$ of the points of P. See Figure 1. The set P is made up of three groups, each group consisting of three subsets arranged in a triangular shape. Each of the nine subsets, call them $1, 2, \ldots, 9$, lies in some disk of sufficiently small diameter δ and contains $\frac{n}{9}$ points.

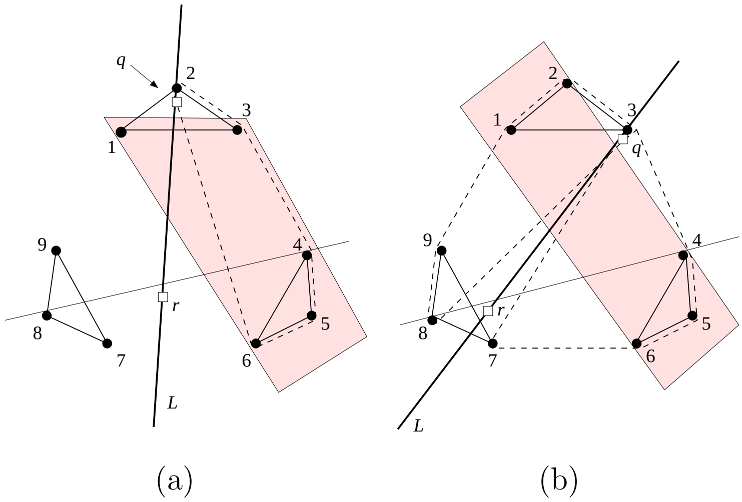

Fig. 1. Lower bound construction for $\varepsilon_2^{\mathcal{C}}$.

For any choice of q and r, let L be the line through q and r. By construction of P, line L can intersect the convex hull of at most two of the subsets $1, \ldots, 9$. We may assume that L has at least three out of the nine subsets fully contained on each side and that L intersects the convex hull of at least one subset. Otherwise, one of the open half-planes bounded by L contains at least $\frac{5}{9}n$

points of P. Write $\mathrm{CH}(i, j, \ldots)$ for the convex hull of the point set $i \cup j \cup \ldots$. Without loss of generality, assume that L intersects $\mathrm{CH}(1, 2, 3)$. We consider two cases:

Case (1): Line L intersects $\mathrm{CH}(2)$; see Figure 1(a). Exploiting symmetries, it is no loss of generality to assume that L be not closer to 6 than to 7. Then, in order to stab $\mathrm{CH}(4, 5, 6, 7, 8)$, one of the points in question, say r, has to lie on or below the upper tangent of $\mathrm{CH}(4)$ and $\mathrm{CH}(8)$. Since we must have $q \in L \cap \mathrm{CH}(2, 3, 4, 5, 6)$, q must lie arbitrarily close to $\mathrm{CH}(2)$ as the disk we assumed to contain the set 2 becomes arbitrarily small. Therefore, for sufficiently small disk diameter δ, $K = \mathrm{CH}(1, 3, 4, 5, 6)$ will avoid both q and r.

Case (2): Line L intersects $\mathrm{CH}(3)$ (or, symmetrically, $\mathrm{CH}(1)$); see Figure 1(b). Again, in order to stab $\mathrm{CH}(4, 5, 6, 7, 8)$, one of the points in question, say r, has to lie on or below the upper tangent of $\mathrm{CH}(4)$ and $\mathrm{CH}(8)$. If L is not closer to 8 than to 7, then we need $q \in L \cap \mathrm{CH}(1, 2, 3, 8, 9)$. Otherwise, we need $q \in L \cap \mathrm{CH}(3, 4, 5, 6, 7)$. In both cases, q must lie arbitrarily close to $\mathrm{CH}(3)$ if δ is chosen to be sufficiently small. Therefore $K = \mathrm{CH}(1, 2, 4, 5, 6)$ avoids both q and r.

To summarize, for any two given points, we can find a convex set K that avoids both points and satisfies $|K \cap P| \geq \frac{5}{9}n$.

In order to derive a lower bound for $\varepsilon_3^{\mathcal{C}}$, we examine the construction above at a higher level. We needed a "tangent condition" for the point r and a "closeness condition" for the point q; refer to Figure 1 again. We now place 4 triangular shaped groups (instead of the three) in a circular manner, each group consisting of three subsets of $\frac{n}{12}$ points of P. This gives $\binom{4}{3} = 4$ instances of the type before. Thus we need to satisfy four tangent conditions as well as four closeness conditions. Two points suffice to satisfy all the tangent conditions plus two closeness conditions. However, the third allowed point cannot be placed to satisfy the two closeness conditions for the remaining two groups simultaneously. We conclude a lower bound of $\frac{5}{12}$ for $\varepsilon_3^{\mathcal{C}}$. □

We now turn to upper bounds. For arbitrary sets of n points in the plane, we want to construct weak ε-nets of given size i and with "deficiency" ε as small as possible. The tools we use are ham-sandwich cuts [8] and weak ε-nets of size at most $i - 1$ that we will have already shown to exist (starting with $i = 1$, the centerpoint discussed in the introduction).

Theorem 3.2 $\varepsilon_2^{\mathcal{C}} \leq \frac{5}{8}$, $\varepsilon_3^{\mathcal{C}} \leq \frac{7}{12}$, $\varepsilon_4^{\mathcal{C}} \leq \frac{4}{7}$, *and* $\varepsilon_5^{\mathcal{C}} \leq \frac{1}{2}$. *For any n-point set,*

[8] For any two sets of points in the plane, a *ham-sandwich cut* is a line that simultaneously splits both sets into two equal parts. Such a line is known to exist (see, e.g., [13]) and can be computed in $O(n)$ time [12].

weak ε-nets with those bounds can be constructed in $O(n)$ time.

Proof: Let P be any n-point set in the plane. Let ℓ be a vertical line that splits the set P into two subsets of, say, r *red* points and b *blue* points. Let h be the ham-sandwich cut line bisecting both the blue and red point sets. Finally, define $q_0 = \ell \cap h$. Refer to Figure 2 where the constructions described below are shown schematically. The point q_0 is indicated by a square marker.

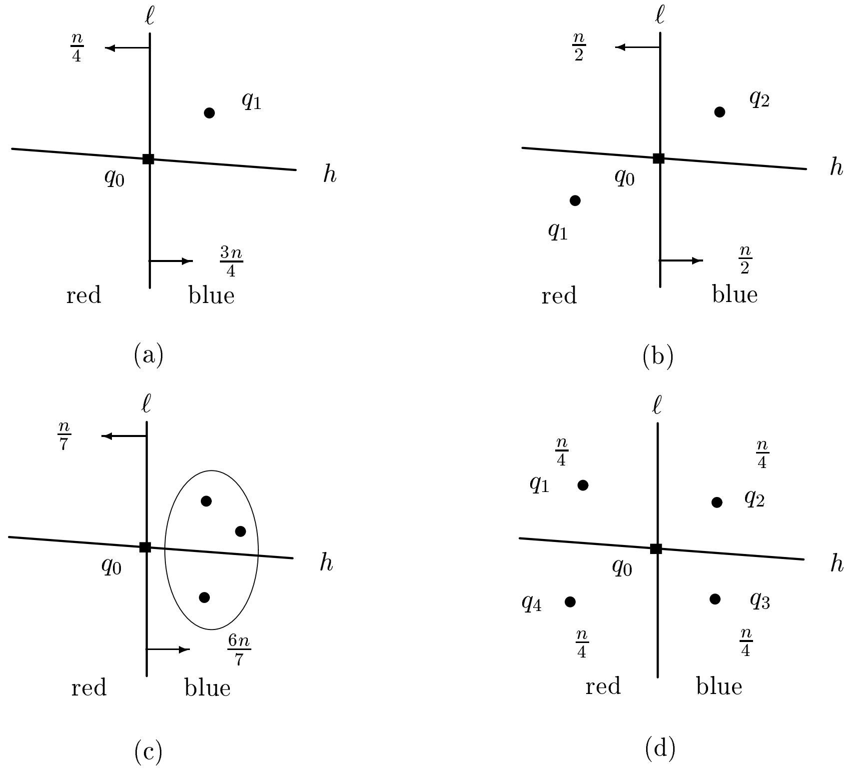

Fig. 2. Small weak ε-nets for convex sets.

We first prove $\varepsilon_2^{\mathcal{C}} \leq \frac{5}{8}$ (see Figure 2(a)). Choose the vertical line ℓ such that $r = \frac{n}{4}$ (and thus $b = \frac{3n}{4}$). Let q_1 be a centerpoint for the blue subset of P. We claim that the set $\{q_0, q_1\}$ is a weak $\frac{5}{8}$-net for P.

Let K be any convex set with $q_0, q_1 \notin K$. As $q_0 \notin K$, the set K avoids at least one of the four quadrants defined by the line ℓ and the ham-sandwich line h. (By convexity, K would contain q_0, otherwise.) If this quadrant is blue then K avoids at least $\frac{3}{8}n$ (blue) points. If this quadrant is red then K avoids at least $\frac{1}{8}n$ red points. In addition, as $q_1 \notin K$, and q_1 is a centerpoint for the blue points, K also avoids at least $\frac{1}{3} \cdot \frac{3n}{4} = \frac{1}{4}n$ blue points. Altogether, K avoids at least $\frac{3}{8}n$ points again. So, in either case, K cannot contain more than $\frac{5}{8}n$ points of P.

Next, we show $\varepsilon_3^{\mathcal{C}} \leq \frac{7}{12}$ (see Figure 2(b)). To this end, choose line ℓ such that $r = \frac{n}{2}$. Then each of the quadrants defined by the lines ℓ and h contains $\frac{n}{4}$ points of P. Take q_1 as a centerpoint for the red points, and take q_2 as a centerpoint for the blue points. Put $Q = \{q_0, q_1, q_2\}$. We argue that Q is a weak $\frac{7}{12}$-net for P.

Let K be any convex set that avoids Q. Since K does not contain q_0, it must avoid some quadrant. Assume, without loss of generality, that this quadrant is blue. Then K can contain at most $\frac{n}{4}$ blue points. Since K avoids q_1, and q_1 is a centerpoint for the red points, K contains at most $\frac{2}{3} \cdot \frac{n}{2}$ red points. In total, at most $(\frac{1}{3} + \frac{1}{4})n = \frac{7}{12}n$ points of P can lie in K.

To see that $\varepsilon_4^{\mathcal{C}} \leq \frac{4}{7}$ holds (see Figure 2(c)), we proceed as follows. Choose $r = \frac{n}{7}$, which gives $b = \frac{6n}{7}$. Let Q_b be a 3-point weak $\frac{7}{12}$-net for the blue points, whose existence we argued above. We claim that the set $Q = Q_b \cup \{q_0\}$ is a weak $\frac{4}{7}$-net for P.

If a convex set K avoids Q then it avoids one quadrant defined by ℓ and h. If it is a blue quadrant then K contains at most $\frac{4}{7}n$ points of P. If this quadrant is red then K contain at most $(\frac{1}{14} + \frac{7}{12} \cdot \frac{6}{7})n = \frac{4}{7}n$ points of P as well, since K contains at most one red quadrant and at most $\frac{7}{12}$ of the blue points.

Finally, we argue that $\varepsilon_5^{\mathcal{C}} \leq \frac{1}{2}$ (see Figure 2(d)). As done for the net of size three, we choose ℓ and h such that each quadrant contains $\frac{n}{4}$ points of P. For the corresponding four subsets P_j of P, let q_j be a centerpoint, for $1 \leq j \leq 4$. Then $Q = \{q_0, \ldots, q_4\}$ is a weak $\frac{1}{2}$-net for P.

Each convex set K that avoids Q totally avoids one subset, say P_1. In addition, K avoids $\frac{1}{3}$ of the points in each of P_2, P_3, and P_4, because the centerpoints of these subsets are in Q. Thus at least $(\frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4})n = \frac{n}{2}$ points are avoided by K.

Centerpoints and ham-sandwich cuts can both be computed in $O(n)$ time [10,12]. Since these tools are used only a constant number of times, the weak ε-nets can be computed in $O(n)$ time. □

Note that there exist other possibilities of combining ham-sandwich cuts with weak ε-nets. For example, when constructing a weak ε-net of size 3, we could use a weak $\frac{5}{8}$-net of size 2 rather than two centerpoints. Then the best vertical cut, $r = \frac{n}{5}$, yields $\varepsilon = \frac{3}{5}$, which is slightly worse than $\varepsilon = \frac{7}{12}$ obtained in the above proof. For weak ε-nets of size four, an alternative way to achieve the same value of ε is to use one centerpoint and a size-2 net. For size 5, no other construction we know of competes with $\varepsilon = \frac{1}{2}$.

In general, it is preferable to use nets that are as small as possible. To obtain an upper bound on $\varepsilon_i^{\mathcal{C}}$ for arbitrary net size i, we may apply the construction for $\varepsilon_5^{\mathcal{C}}$ recursively. This evaluates to

$$\varepsilon_i^{\mathcal{C}} \leq \frac{2}{3}\left(\frac{3}{4}\right)^k, \text{ for } i = \frac{1}{3}(4^{k+1}-1), \quad k \geq 0.$$

A rough calculation shows that a weak ε-net of size $O(\frac{1}{\varepsilon^5})$ with respect to $\mathcal{C}$ is obtained. Unfortunately (but not surprisingly) this falls far short of the best known bound $O(\frac{1}{\varepsilon^2})$ in [1]; see also [5]. Still, for small nets, our constructions are superior. For example, to achieve $\varepsilon = \frac{1}{2}$ a net of size eight (rather than five) is needed in [1,5].

4 Axis-parallel rectangles

This section presents bounds on $\varepsilon_i^{\mathcal{R}}$, where $\mathcal{R}$ denotes the family of all axis-parallel rectangles in the plane. All proofs assume that the point set P is in general position in the sense that no two points have the same x- or y-coordinate. This assumption can be removed by symbolically perturbing the point set, and observing that a weak ε-net Q for the perturbed set is a weak ε-net for the original set P. To see this, suppose for a contradiction there is a rectangle R that contains more than εn points of P and avoids Q. Since R is a closed rectangle, it is possible to expand it slightly so that it captures no more points and the coordinates of its corners does not coincide with those of any point of $P \cup Q$. Thus, the set of points contained in that rectangle in the perturbed set is no different than in the original set. Thus Q is not a weak ε-net for the perturbed set, a contradiction.

Theorem 4.1 *$\varepsilon_1^{\mathcal{R}} \geq \frac{1}{2}$, $\varepsilon_2^{\mathcal{R}} \geq \frac{2}{5}$, and $\varepsilon_3^{\mathcal{R}} \geq \frac{1}{3}$.*

Proof: Let P be any set of n points and let q be some point. One of the open half-planes bounded by the vertical line through q contains at least $\lfloor (n-1)/2 \rfloor \geq n/2-2$ points of P. In addition, there exists an axis-parallel rectangle enclosing those points and avoiding q. Thus $\varepsilon_1^{\mathcal{R}} \geq 1/2-2/n$. As n can be chosen to be arbitrarily large, this implies $\varepsilon_1^{\mathcal{R}} \geq \frac{1}{2}$.

Next, we show $\varepsilon_2^{\mathcal{R}} \geq \frac{2}{5}$. Suppose, for a contradiction, that $\varepsilon_2^{\mathcal{R}} = \varepsilon < \frac{2}{5}$. Let h_1, h_2 be two horizontal lines and v_1, v_2 two vertical lines, with h_1 above h_2 and v_1 left of v_2. These four lines partition the plane into 9 axis-parallel rectangles (some unbounded). Denote those rectangles by $A_{i,j}$ for $i, j = 1, \ldots, 3$, where $A_{i,j}$ is the rectangle defined by the ith row and the jth column. Let n be a multiple of 5 and place $\lfloor n/5 \rfloor$ points in each of the rectangles $A_{1,1}, A_{1,3}, A_{2,2}, A_{3,1}, A_{3,3}$. See Figure 3. If a pair of points $Q = \{q_1, q_2\}$ is a weak ε-net for P with respect to axis-parallel rectangles and $\varepsilon < \frac{2}{5}$, then each of the four strips above h_1, below h_2, left of v_1 and right of v_2 must contain a point of Q. Since no triple of strips has a common intersection, each of the 2 points must be contained in exactly two strips. Then either $Q \subset A_{1,3} \cup A_{3,1}$ or $Q \subset A_{1,1} \cup A_{3,3}$. Assume without loss of generality the former case. Then $A_{1,1} \cup A_{1,2} \cup A_{2,1} \cup A_{2,2}$ is an axis-parallel rectangle containing $\frac{2}{5}n$ points of P and avoiding $\{q_1, q_2\}$, a contradiction.

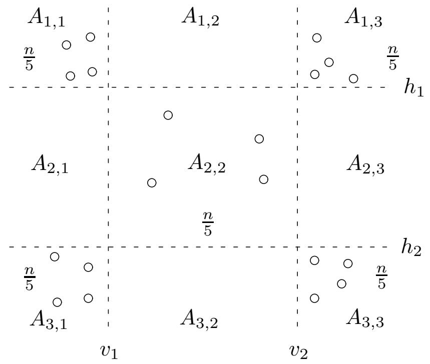

Fig. 3. A lower bound construction showing that $\varepsilon_2^{\mathcal{R}} \geq \frac{2}{5}$.

We next show the lower bound $\varepsilon_3^{\mathcal{R}} \geq \frac{1}{3}$. For a contradiction, assume $\varepsilon_3^{\mathcal{R}} = \varepsilon < \frac{1}{3}$. Define h_1, h_2, v_1, v_2 as in the previous case. For n a multiple of 6, place $n/6$ points in each of $A_{1,1}, A_{1,3}, A_{3,1}, A_{3,3}$, and the remaining $n/3$ points in $A_{2,2}$. See Figure 4 for illustration. Suppose Q is a weak ε-net. Since $A_{2,2}$ contains $n/3$ points, it must contain a point of Q. As before, the four extreme strips above h_1, below h_2, left of v_1 and right of v_2 must each contain some point of Q. This implies that either there is a point in each of the rectangles $A_{i,i}$ or there is a point in each of the rectangles $A_{i,4-i}$ for $i = 1, \ldots, 3$. Assume the latter case without loss of generality. Consider the point q which lies in the rectangle $A_{2,2}$. Since $A_{2,2}$ contains $\frac{n}{3}$ points, either the region of $A_{2,2}$ above q or the region below q contains at least $n/6 - 1$ points. This region along with one of the corners $A_{1,1}$ or $A_{3,3}$ would determine a rectangle which contained at least $n/3 - 1$ points of P and no points of Q. Since, for large enough n, $1/3 - 1/n > \varepsilon$, this contradicts the assumption that Q is a weak ε-net.

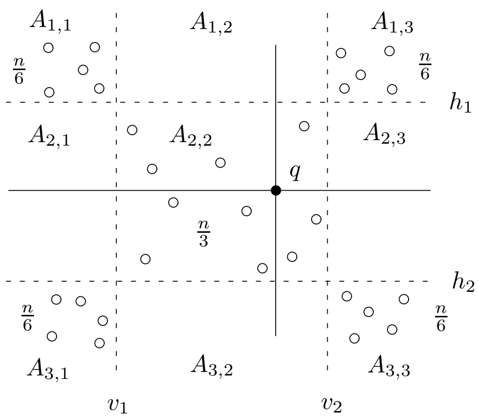

Fig. 4. A lower bound construction showing that $\varepsilon_3^{\mathcal{R}} \geq \frac{1}{3}$.

□

Lemma 4.2 *For all positive integers k, i, j, and $\ell \le k+1$,*

$$\varepsilon^{\mathcal{R}}_{k^2+2\ell i+2(k+1-\ell)j} \le \frac{\varepsilon^{\mathcal{R}}_i \varepsilon^{\mathcal{R}}_j}{\ell\varepsilon^{\mathcal{R}}_j + (k+1-\ell)\varepsilon^{\mathcal{R}}_i}.$$

Proof: Let $h_1, \ldots, h_k$ be k horizontal lines dividing the plane into $k+1$ horizontal strips, such that ℓ of these open strips contain at most δn points, and the other $k+1-\ell$ contain at most γn points, with $\ell\delta+(k+1-\ell)\gamma = 1$ (we say "at most" and not "exactly" to take care of rounding and points incident to the lines h_i). Likewise, let $v_1, \ldots, v_k$ be k vertical lines splitting the plane into vertical strips containing points in the same proportions. We construct the net Q using the grid of $k \times k$ points at the intersection of all pairs of lines. In each of the 2ℓ strips (horizontal or vertical) with at most δn points, we add to Q a $\varepsilon^{\mathcal{R}}_i$-net using i points for the points of P in that strip. Likewise, we add to Q a $\varepsilon^{\mathcal{R}}_j$-net using j points for each of the $2(k+1-\ell)$ remaining strips. Thus, $|Q| = k^2 + 2\ell i + 2(k+1-\ell)j$. We choose δ and γ so that $\delta\varepsilon^{\mathcal{R}}_i = \gamma\varepsilon^{\mathcal{R}}_j$. Solving this system of equations for δ and γ, we find that

$$\delta = \frac{\varepsilon^{\mathcal{R}}_j}{\ell\varepsilon^{\mathcal{R}}_j + (k+1-\ell)\varepsilon^{\mathcal{R}}_i}, \text{ and}$$
$$\gamma = \frac{\varepsilon^{\mathcal{R}}_i}{\ell\varepsilon^{\mathcal{R}}_j + (k+1-\ell)\varepsilon^{\mathcal{R}}_i}.$$

Thus Q forms the desired weak ε-net. □

Theorem 4.3 $\varepsilon^{\mathcal{R}}_2 \le \frac{2}{5}$.

Proof: Let v_1 be a vertical line with exactly $\frac{2}{5}n$ points of P to its left and let v_2 be a vertical line with exactly $\frac{2}{5}n$ points of P to its right. Similarly consider a line h_1 (resp., h_2) with exactly $\frac{2}{5}n$ points of P above it (resp., below it). Let q_1, q_2, q_3, q_4 be the vertices of the rectangle formed by the intersection points of these lines; refer to Figure 5. Observe that the set $\{q_1, q_2, q_3, q_4\}$ is a weak $\frac{2}{5}$ net for P. We will show that at least one of the sets $Q_1 = \{q_1, q_3\}$, $Q_2 = \{q_2, q_4\}$ is a weak $\frac{2}{5}$-net for P. Assume to the contrary that neither of these sets is a weak $\frac{2}{5}$-net for P. Thus since Q_1 is not a desired net, there exists a rectangle avoiding the set Q_1 and containing more than $\frac{2}{5}n$ points of P. Such a rectangle must contain either q_2 or q_4. Assume without loss of generality that the later situation occurs. Symmetrically, there must exists a rectangle proving that Q_2 is not a weak $\frac{2}{5}$-net, i.e., containing at least $\frac{2}{5}n$ points of P and, say, q_1. Let A, B, C, D, E, F be the number of points in each of the six rectangles induced by the arrangement of the lines v_1, v_2, h_1, h_2 that lie below h_1, as indicated in Figure 5. We have

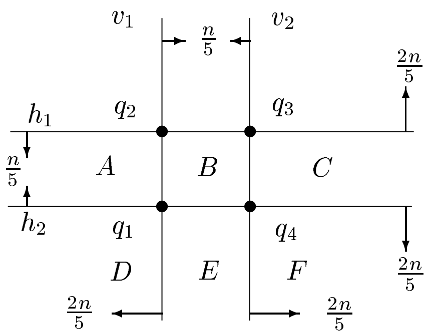

Fig. 5. The vertical lines v_1, v_2 and horizontal lines h_1, h_2 intersect in four points q_1, q_2, q_3, q_4.

$$A + B + C = \frac{n}{5}, \qquad B + C + E + F > \frac{2n}{5},$$
$$D + E + F = \frac{2n}{5}, \text{ and} \qquad A + B + D + E > \frac{2n}{5}.$$

Summing the two inequalities and subtracting the two equalities we get $B + E > \frac{n}{5}$, a contradiction. Hence, $\varepsilon_2^{\mathcal{R}} \leq \frac{2}{5}$. □

Theorem 4.4 $\varepsilon_1^{\mathcal{R}} \leq \frac{1}{2}$, $\varepsilon_3^{\mathcal{R}} \leq \frac{1}{3}$, $\varepsilon_5^{\mathcal{R}} \leq \frac{1}{4}$, $\varepsilon_7^{\mathcal{R}} \leq \frac{2}{9}$, $\varepsilon_8^{\mathcal{R}} \leq \frac{1}{5}$, $\varepsilon_{10}^{\mathcal{R}} \leq \frac{1}{6}$, $\varepsilon_{16}^{\mathcal{R}} \leq \frac{2}{15}$.

Proof: Those bounds follow from the fact that $\varepsilon_0^{\mathcal{R}} = 1$ and lemma 4.2, with $k = 1$, $l = 2$, $i = 0$, $j = 0$ for $\varepsilon_1^{\mathcal{R}}$; $k = 1$, $l = 1$, $i = 0$, $j = 1$ for $\varepsilon_3^{\mathcal{R}}$; $k = 1$, $l = 2$, $i = 1$, $j = 0$ for $\varepsilon_5^{\mathcal{R}}$; $k = 1$, $l = 1$, $i = 1$, $j = 2$ for $\varepsilon_7^{\mathcal{R}}$; $k = 2$, $l = 1$, $i = 0$, $j = 1$ for $\varepsilon_8^{\mathcal{R}}$; $k = 2$, $l = 3$, $i = 1$, $j = 0$ for $\varepsilon_{10}^{\mathcal{R}}$; and $k = 2$, $l = 3$, $i = 2$, $j = 0$ for $\varepsilon_{16}^{\mathcal{R}}$. □

Lemma 4.5 $\varepsilon_4^{\mathcal{R}} \leq \frac{5}{16}$.

Proof: Let v_1 be a vertical line with exactly δn points of P to its left and let v_2 be a vertical line with exactly δn points of P to its right. Similarly consider a line h_1 (resp., h_2) with exactly δn points of P above it (resp., below it). The four lines h_1, h_2, v_1, v_2 partition the plane into 9 axis-parallel rectangles. Denote the proportion of points in those rectangles by $A_{i,j}$ for $i, j = 1, \ldots, 3$, where $A_{i,j}$ corresponds to the rectangle defined by the ith row and the jth column. Summing the number of points left of v_1, right of v_2, above h_1 and below h_2, we get:

$$4\delta = 1 + A_{1,1} + A_{1,3} + A_{3,1} + A_{3,3} - A_{2,2}.$$

Thus, assuming (without loss of generality) that $A_{1,1}$ is the corner containing the most points, we get $A_{1,1} \geq \frac{4\delta - 1}{4}$. Pick $v_1 \cap h_1$ as the first point in the net, and construct a 1/3-net for the $n(1 - \frac{4\delta-1}{4})$ points in $P - A_{1,1}$, using the remaining 3 points. Setting

$$\delta = \frac{1}{3}(1 - \frac{4\delta - 1}{4})$$

yields $\delta = \frac{5}{16}$. □

	$\varepsilon_1^{\mathcal{R}}$	$\varepsilon_2^{\mathcal{R}}$	$\varepsilon_3^{\mathcal{R}}$	$\varepsilon_4^{\mathcal{R}}$	$\varepsilon_5^{\mathcal{R}}$	$\varepsilon_7^{\mathcal{R}}$	$\varepsilon_8^{\mathcal{R}}$	$\varepsilon_{10}^{\mathcal{R}}$	$\varepsilon_{16}^{\mathcal{R}}$
Upper bounds	1/2	2/5	1/3	5/16	1/4	2/9	1/5	1/6	2/15
Lower bounds	1/2	2/5	1/3	1/5	1/6	1/8	1/9	1/11	1/17

Table 2
Bounds for axis-parallel rectangles. Lower bounds on $\varepsilon_i^{\mathcal{R}}$, $i \geq 4$ follow from Lemma 2.1.

5 Remarks

It is interesting to note that some bounds on the size of weak ε-nets follow rather directly from classical results. We illustrate this fact for the collection $\mathcal{D}$ of all disks in the plane.

Theorem 5.1 $\varepsilon_4^{\mathcal{D}} \leq \frac{1}{2}$.

Proof: Let P be a set of n points in the plane. We need to show that there exists a set Q of four points such that every disk d for which $|d \cap P| > \frac{n}{2}$ must intersect Q. Consider the collection $\mathcal{D}' \subset \mathcal{D}$ of all disks d that contain more than $\frac{n}{2}$ points of P. Obviously every pair of disks of $\mathcal{D}'$ must have a non-empty intersection. By the result of [6], there exists a set Q of four points that stab all disks in $\mathcal{D}'$. This completes the proof. □

In [8] it was proved that for any finite collection of pairwise intersecting unit disks, there exists three points that stab those disks. Thus, using the same analysis as in the proof above we have that $\varepsilon_3^{\mathcal{U}} \leq \frac{1}{2}$, where $\mathcal{U}$ is the collection of all unit disks in the plane.

Several papers have appeared on the topic of small weak ε-nets since the preliminary version of this paper was published [3]. In [4], the authors use a generalized ham-sandwich cut to show that $e_4^{\mathcal{C}} \leq 6/11$. They then provide several bounds for halfspaces and convex sets in $\mathbb{R}^3$. In [14], improved bounds are shown for convex sets in $\mathbb{R}^2$: $\varepsilon_2^{\mathcal{C}} = 4/7$, $5/11 \leq \varepsilon_3^{\mathcal{C}} \leq 8/15$, $\varepsilon_4^{\mathcal{C}} \leq 16/31$ and $\varepsilon_5^{\mathcal{C}} \leq 20/41$. In an undergraduate thesis, Dulieu [7] proved that $\varepsilon_4^{\mathcal{R}} \leq 2/7$ using a case analysis. This shows that $\varepsilon_i^{\mathcal{R}} \leq 2/(i+3)$ for $1 \leq i \leq 5$. It would be interesting to prove that this is indeed true for all i.

References

[1] N. Alon, I. Bárány, Z. Füredi, and D. J. Kleitman. Point selections and weak ε-nets for convex hulls. *Combinatorics, Probability & Computing*, 1:189–200, 1992.

[2] N. Alon, H. Kaplan, G. Nivasch, M. Sharir, and S. Smorodinsky. Weak ε-nets and interval chains. *JACM, accepted*, 2008. Also appears in Proc. of 19th ACM-SIAM Symposium on Discrete Algorithms (SODA 2008).

[3] B. Aronov, F. Aurenhammer, F. Hurtado, S. Langerman, D. Rappaport, C. Seara, and S. Smorodinsky. Small weak epsilon nets. In *Proceedings of the 17th Canadian Conference on Computational Geometry (CCCG 2005)*, pages 52–56, 2005.

[4] M. Babazadeh and H. Zarrabi-Zadeh. Small weak epsilon-nets in three dimensions. In *Proceedings of the 18th Canadian Conference on Computational Geometry (CCCG 2006)*, pages 47–50, 2006.

[5] B. Chazelle. *The discrepancy method: randomness and complexity.* Cambridge University Press, New York, NY, USA, 2000.

[6] L. Danzer. Zur Lösung des Gallaischen Problems über Kreisscheiben in der euklidischen Ebene. *Studia Scientiarum Mathematicarum Hungarica*, 21:111–134, 1986.

[7] M. Dulieu. ε-nets faibles. Mémoire de Licence, Université Libre de Bruxelles, June 2006.

[8] H. Hadwiger, H. Debrunner, and V. Klee. *Combinatorial Geometry in the Plane.* Holt, Rinehart & Winston, New York, 1964.

[9] D. Haussler and E. Welzl. Epsilon-nets and simplex range queries. *Discrete Comput. Geom.*, 2:127–151, 1987.

[10] S. Jadhav and A. Mukhopadhyay. Computing a centerpoint of a finite planar set of points in linear time. *Discrete Comput. Geom.*, 12:291–312, 1994.

[11] J. Komlós, J. Pach, and G. Woeginger. Almost tight bounds for ε-nets. *Discrete Comput. Geom.*, 7:163–173, 1992.

[12] C.-Y. Lo, J. Matoušek, and W. L. Steiger. Algorithms for ham-sandwich cuts. *Discrete Comput. Geom.*, 11:433–452, 1994.

[13] J. Matousek. *Lectures on Discrete Geometry.* Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2002.

[14] N. Mustafa and S. Ray. An optimal generalization of the centerpoint theorem, and its extensions. In *SCG '07: Proceedings of the Twenty-Third Annual Symposium on Computational Geometry*, pages 138–141, 2007.