Some Lower Bounds on Geometric Separability Problems *

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Abstract

In this paper we obtain lower bounds in the algebraic computation tree model for deciding the separability of two disjoint point sets. More precisely, we show $\Omega(n \log n)$ time lower bounds for the separability by means of strips, wedges, wedges with apices on a given line, fixed-slopes double wedges, and triangles (solving an open problem from Edelsbrunner and Preparata), which match the complexity of the existing algorithms, and therefore prove their optimality.

1 Introduction

Let B and R be two finite disjoint sets of n points in the plane; we refer to B as the set of *blue* points and R as the set of *red* points. A finite set S of curves in the plane is a *separator* for the sets B and R if every connected component in $\mathbb{R}^2 \setminus S$ contains objects only from B or only from R. We also say that each connected component is *monochromatic*. Separability can also be defined for sets of objects other than points and for objects in higher dimensions. It can also be extended to cases in which strict separability is not possible, as suggested by Houle [13]. Decision and optimization problems in this area have attracted much attention.

The most natural notion of separability in the plane is by means of a line; when that is possible we say that B and R are *line separable*. It is well known that the decision problem of linear separability for sets of points, segments or circles can be solved in $\Theta(n)$ time [13, 17].

Bhathacharya, Boissonnat et al., O'Rourke et al. and Fish have studied the case in which S is a circle (the *circular separability problem*), considering also the problem of finding the smallest and largest such separators [4, 5, 9, 18]. Edelsbrunner and Preparata [7] considered the problem of convex polygonal separators, giving algorithms both for the decision problem and the problem of minimizing the number of edges in the separator.

A *wedge* is the union of two rays with common origin (the *apex*) (Figure 1a). A *strip* is the union of two parallel lines; their common slope is the *slope of the strip* (Figure 1b). Algorithms for deciding wedge separability and strip separability, as well as for constructing the locus of feasible

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apices and the interval of feasible slopes, are described in [14]; it is also shown how to find wedges with maximum/minimum angle, and the narrowest/widest strip. A *double wedge* is the union of two crossing lines (Figure 1c). Efficient algorithms for deciding the existence of a double wedge separator are given in [15]. All those algorithms have $O(n \log n)$ running time.

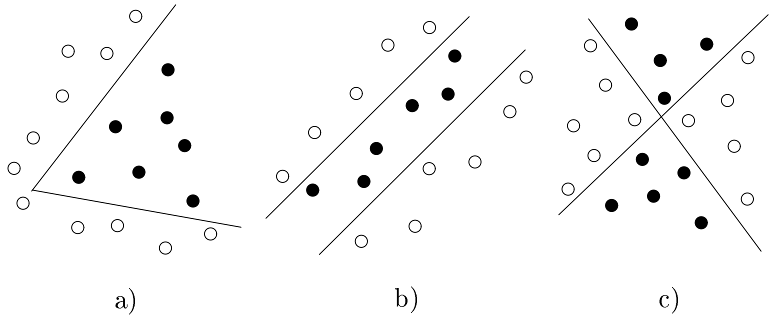

Figure 1: a) Wedge, b) strip, c) double wedge.

Other related problems include the minimum-cardinality *shattering* problem (find an arrangement of the fewest lines whose arrangement decomposes the plane into monochromatic cells [11]), and the *minimum-link red-blue separation problem* (find a polygonal separator with fewest edges that separate B from R). Both problems are known to be NP-complete [8, 10, 11].

In this paper we show $\Omega(n \log n)$ time lower bounds for: (1) deciding strip separability, including cases in which we restrict the strip to pass through one or two given points or in which we fix the width of the strip; (2) deciding wedge separability, including cases in which we restrict the wedge to pass through one or two given points, restricting the apices of the wedges to lie on a given line or segment and, a half-line of the wedge having a given slope; (3) deciding triangle separability, including cases in which we restrict the vertices of the triangle to lie on given lines, only a line containing one side is given, one or two vertices of the triangle are given and, two lines supporting sides are given; and, (4) deciding fixed-slopes double wedge separability, including the case in which we restrict the apex of the double wedge to lie on a given line. It is worth mention that all these bounds easy extend to minimizing the number of points misclassified by a wedge or by a strip;

Summarizing these results, we see that when we jump from linear separability (one line) to separability involving *two* lines (including also some additional information), we have to pay a $\log n$ factor in the complexity of the algorithms for the decision problems. All of our constructions are based on a lower bound for some problems on gaps defined by points, which we prove first.

After a section on the basic problems for the reductions we give lower bounds for the separability by means of strips, wedges, triangles and double wedges in Section 3, 4, 5 and 6, respectively, and we conclude with some observations and a summary table in Section 7.

2 MAX-GAP Problems

Given a set $S = \{x_1, \ldots, x_n\}$ of n real numbers, throughout this paper we denote by $x_{S_1} \leq \cdots \leq x_{S_n}$ the sequence of these numbers once sorted. The *maximum gap* of S is defined to be the maximum difference between two consecutive members of S: $\text{MAX-GAP}(S) = \max_i \{x_{S_i} - x_{S_{i-1}}\}$. This value is easily computed after sorting, and this computation has an $\Omega(n \log n)$ lower bound in the algebraic decision tree model, as proved by Lee and Wu [16].

Given a set $S = \{x_1, \dots, x_n\}$ of n real numbers and a positive real number ϵ, the problem of deciding whether MAX-GAP$(S) > \epsilon$ also has an $\Omega(n \log n)$ lower bound in the same model, as proved by Ramanan [19], who introduces the technique of using *artificial components*. Unfortunately, none of the proofs of the above problems extends in the algebraic computation tree model to the following variation:

Problem Greater-or-Equal (GE) . *Given a set $S = \{x_1, \dots, x_n\}$ of n real numbers and a positive real number $\epsilon \in \mathbb{R}^+$, determine whether* MAX-GAP$(S) \geq \epsilon$.

A basic result in this paper is an $\Omega(n \log n)$ lower bound for problem GE, which is then easily modified for proving the same bound for the following problem:

Problem Quadrant-Greater-or-Equal (QGE) . *Given n points $\{(x_1, y_1), \dots, (x_n, y_n)\}$ in the first quadrant on the unit circle and a positive real number $\epsilon \in \mathbb{R}^+$, decide whether the Euclidean distance between some two points that are consecutive along the circle is $\geq \epsilon$.*

Most of our $\Omega(n \log n)$ lower bound constructions in the next sections use a reduction from problem QGE or from problem GE. The decision problem GE is trivially equivalent to its complement:

Problem Complement-Greater-or-Equal (CGE) . *Given a set $S = \{x_1, \dots, x_n\}$ of n real numbers and a positive real number $\epsilon \in \mathbb{R}^+$, decide whether* MAX-GAP$(S) < \epsilon$.

Problems GE and CGE are also trivially equivalent to the following problem, since the union of the intervals is connected if, and only if, MAX-GAP$\{x_1, \dots, x_n\} < \epsilon$:

Problem Connected-Union (CU) . *Given a set $S = \{x_1, \dots, x_n\}$ of n real numbers and a positive real number $\epsilon \in \mathbb{R}^+$, determine whether the union of the intervals*

$$(x_1 - \frac{\epsilon}{2}, x_1 + \frac{\epsilon}{2}), \dots, (x_n - \frac{\epsilon}{2}, x_n + \frac{\epsilon}{2})$$

is connected.

Now we consider an auxiliary problem:

Problem Auxiliary (AUX) . *Given a set $S = \{x_1, \dots, x_n\}$ of n real numbers and a positive real number $\epsilon \in \mathbb{R}^+$, decide whether* MAX-GAP$\{x_1, \dots, x_n, 0, \epsilon, 2\epsilon, \dots, n\epsilon\} < \epsilon$.

Observe that any algorithm which solves CGE also solves AUX. Now we prove an $\Omega(n \log n)$ lower bound for AUX.

Theorem 1. *Given a set $S = \{x_1, \dots, x_n\}$ of n real numbers and a positive real number $\epsilon \in \mathbb{R}^+$, deciding whether*

$$\text{MAX-GAP}\{x_1, \dots, x_n, 0, \epsilon, 2\epsilon, \dots, n\epsilon\} < \epsilon$$

has an $\Omega(n \log n)$ lower bound in the algebraic computation tree model.

Proof. In order to have MAX-GAP $\{x_1, \dots, x_n, 0, \epsilon, 2\epsilon, \dots, n\epsilon\} < \epsilon$, each of the open intervals $(0, \epsilon)$, $(\epsilon, 2\epsilon)$,...,$((n-1)\epsilon, n\epsilon)$ has to be pierced by one of the x_i's. Now we consider the set $W \subset \mathbb{R}^n$ of points $(x_1, \dots, x_n)$ that correspond to piercing sets for the intervals. Given any point $(x_1, \dots, x_n) \in W$, we know that $(x_{\pi_1}, x_{\pi_2}, \dots, x_{\pi_n})$ is also a point in W for any permutation π of $\{1, 2, \dots, n\}$; however, for distinct permutations of $\{1, 2, \dots, n\}$, we know that the corresponding points of W must lie in distinct connected components of W, since any swap from $(\dots, x_i, \dots, x_j, \dots)$ to $(\dots, x_j, \dots, x_i, \dots)$ requires that some coordinate value pass through at least one of the values $\{\epsilon, 2\epsilon, 3\epsilon, \dots, (n-1)\epsilon\}$. Thus, W has at least $n!$ connected components. By the theorem of Ben-Or [3], $\Omega(\log n!) = \Omega(n \log n)$ is a lower bound in the algebraic computation tree model. □

For the QGE problem the same proof applies: one has to construct points in the first quadrant on the unit circle spaced by Euclidean distance ϵ, starting at $(0, 1)$, which will play the role of the $n \cdot \epsilon$ above. This construction is carried out using only $+, -, *, /$ and $\sqrt{\ }$. The next theorem summarizes our basic results.

Theorem 2. *Each of the problems GE, QGE, CGE, and CU has an $\Omega(n \log n)$ lower bound in the algebraic computation tree model.*

3 Strip Separability

In [14], $O(n \log n)$-time algorithms are given for deciding strip separability of point sets, as well as for constructing the interval of slopes for which the points are strip separable and for computing the narrowest/widest separating strip, if one exists. In this section we prove that these algorithms are worst-case optimal, by showing an $\Omega(n \log n)$ lower bound on deciding strip separability, both in the general case, and in a variety of special cases.

3.1 Strip separability in the general case

Theorem 3. *Deciding whether two disjoint point sets B and R are strip separable requires $\Omega(n \log n)$ operations in the algebraic computation tree model.*

Proof. We reduce QGE to strip separability in linear time. Let $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ and $\epsilon \in \mathbb{R}^+$ be an instance of QGE; S is a set of n points in the first quadrant on the unit circle. We have to decide whether the Euclidean distance between some two consecutive points (the order is given by the x-coordinate) is $\geq \epsilon$.

If $\epsilon \geq \sqrt{2}$, we conclude that it is impossible to have a separation that is $\geq \epsilon$, since the points $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ lie on the unit circle in the first quadrant.

Assume now that $\epsilon < \sqrt{2}$. We define an instance of strip separability for two disjoint point sets R and B defined as follows.

1. We let $R = S \cup S'$, where $S' = \{(-x_1, -y_1), \ldots, (-x_n, -y_n)\}$ is a reflection of S into the third quadrant.

2. We let $B = \{b_1, b_2\} \cup R^\perp$, where

$$R^\perp = \{(-dy_1, dx_1), \ldots, (-dy_n, dx_n)\} \cup \{(dy_1, -dx_1), \ldots, (dy_n, -dx_n)\}$$

   is the set of points R rotated by 90 degrees and scaled by a number $d > 0$, and b_1 (resp., b_2) is the point of intersection between the line through the leftmost two points of S (resp., S') and the line through the rightmost two points of S (resp., S'). Refer to Figure 2.

3. We select d so that a gap of size ϵ between two consecutive points of S determines a line, ℓ, through these two points, and a line, ℓ', through the corresponding two points of S', such that ℓ and ℓ' pass through the corresponding blue points on the circle of radius d. Refer to Figure 2, where we let a denote the distance from the origin to ℓ or ℓ', and we illustrate the similar triangles defined by the consecutive pair of red points and the consecutive pair of blue points. By similar triangles, we see that our choice of d should be $d = 2a/\epsilon = \frac{\sqrt{4-\epsilon^2}}{\epsilon}$. One can readily check that as long as $0 < \epsilon < \sqrt{2}$, $1 < d < +\infty$, confirming that the circle of radius d containing all but two of the blue points is indeed larger than the unit circle, on which the red points lie.

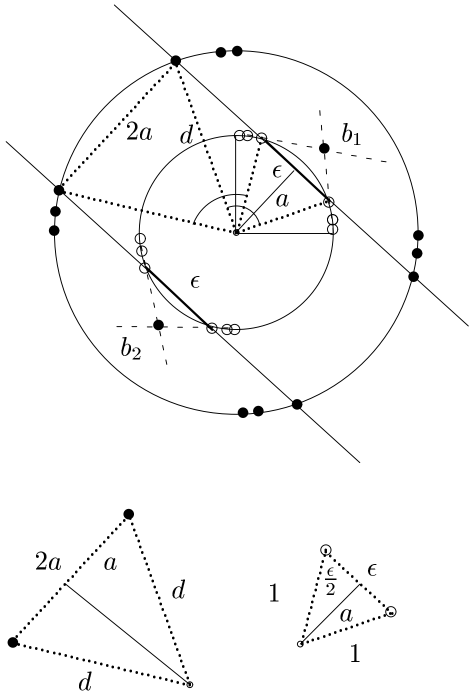

Figure 2: Construction used in proving the lower bound for strip separability in the general case. Red points (shown as hollow dots) lie on a unit-radius circle, while all but two of the blue points (shown as solid dots) lie on a concentric circle of radius d.

If B and R are strip separable, then, by construction, the narrowest separating strip is of width $\geq 2a$ and is determined by two lines: one passing through two consecutive red points in the first quadrant that are at distance $\geq \epsilon$, and one passing through the corresponding red points in the third quadrant (also at distance $\geq \epsilon$, of course). Conversely, if there exists a pair of consecutive red points in the first quadrant at distance $\geq \epsilon$, then the corresponding blue points are separated by at least distance $2a$, and there exists a separating strip. Thus, B and R are strip separable if and only if the Euclidean distance between some two consecutive red points in the first quadrant is $\geq \epsilon$. □

3.2 Strip separability with constraints

In this subsection we show that knowing some additional information does not help to decide strip separability, in that $\Omega(n \log n)$ remains a lower bound in the algebraic computation tree model.

3.2.1 Strip through a given point

We consider first the constraint in which one line of the strip is restricted to pass through a given point.

Theorem 4. *Deciding whether two disjoint point sets B and R are strip separable with the restriction that one line of the strip must pass through a given point has an $\Omega(n \log n)$ lower bound in the algebraic computation tree model.*

Proof. We reduce QGE to strip separability with the restriction that one line of the strip must pass through a given point in linear time. Let $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ and $\epsilon \in \mathbb{R}^+$ be an instance of QGE.

To decide whether the Euclidean distance between some two consecutive points is $\geq \epsilon$, we do the following.

1. If $\epsilon > \sqrt{2}$, we reject, concluding that it is impossible to have a separation that is $\geq \epsilon$, since the points $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ lie on the unit circle in the first quadrant.

2. If $\sqrt{2-\sqrt{2}} \leq \epsilon \leq \sqrt{2}$, then the only way for the Euclidean distance to be at least ϵ between some two consecutive points, (x_i, y_i) and (x_{i+1}, y_{i+1}), is if (x_i, y_i) is in the first half of the first quadrant (i.e., $x_i < y_i$) and (x_{i+1}, y_{i+1}) is in the second half of the first quadrant (i.e., $x_{i+1} > y_{i+1}$), as shown in Figure 3. We readily determine (in linear time) the points (x_i, y_i) and (x_{i+1}, y_{i+1}); e.g., (x_i, y_i) is the point with the largest x-coordinate that is $< \frac{\sqrt{2}}{2}$. We then accept if and only if the (straight-line) distance between (x_i, y_i) and (x_{i+1}, y_{i+1}) is $\geq \epsilon$.

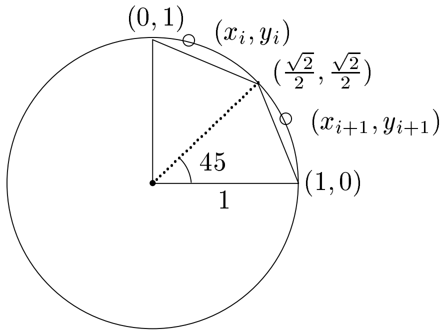

Figure 3: Illustration of the case $\sqrt{2-\sqrt{2}} \leq \epsilon \leq \sqrt{2}$.

In the case that $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ is one of the input points and $\epsilon = \sqrt{2-\sqrt{2}}$, then we check whether $(1,0)$ or $(0,1)$ are points of the input set, and, if so, accept if and only if there are no input points in between $(0,1)$ and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ or in between $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(1,0)$.

3. If $\epsilon < \sqrt{2-\sqrt{2}}$, then, just as we did in the proof of Theorem 3, we construct sets of red and blue points on two concentric circles. The radius of the circle with blue points is chosen to be $d = \frac{1}{\epsilon}$, which we justify below. Also, we do not place red points in the third quadrant, but, instead, we place a red point r in the center of the red circle; point r is the given point through which a line of the strip must pass. Let R be the set of these $n+1$ red points, and let B be the set of the $2n+2$ blue points of the blue circle, as shown in Figure 4.

4. We select d so that a gap of size ϵ between two consecutive red points in the first quadrant determines a line, ℓ, through these two points, and a line parallel to ℓ passing through the center red point, such that the strip they define just separates the pairs of consecutive blue points corresponding to the two red points determining ℓ. Refer to Figure 4, where we let a denote the distance from the origin to ℓ, and we illustrate the similar triangles defined by the consecutive pair of red points and the consecutive pair of blue points. By similar triangles, we see that our choice of d should be $d = 1/\epsilon$. □

3.2.2 Strip through two given points

We now see that even if *two* points are fixed through which a separating strip is required to pass, the lower bound for strip separability remains the same as in the unconstrained case:

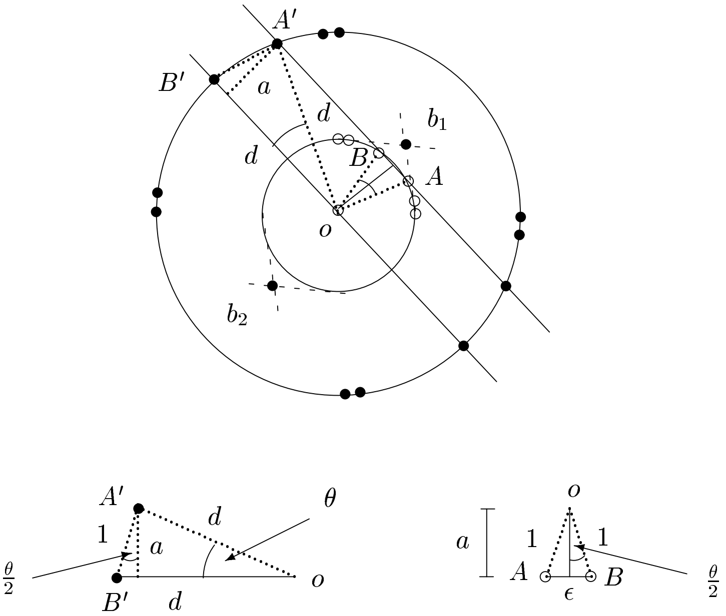

Figure 4: Construction for the case of a strip restricted to have one line pass through a given point, r.

Theorem 5. *Deciding whether two disjoint point sets B and R are strip separable with the restriction that the parallel lines pass through two given points has an $\Omega(n \log n)$ lower bound in the algebraic computation tree model.*

Proof. We give a linear time reduction of GE to strip separability with the restriction that the parallel lines pass through two given points. Let $S = \{x_1, \ldots, x_n\}$ and $\epsilon \in \mathbb{R}^+$ be an instance of GE; we have to determine whether MAX-GAP$(S) \geq \epsilon$. We do the following construction.

1. Represent $\{x_1, \ldots, x_n\}$ by the blue points $\{b_1, \ldots, b_n\}$ on a line ℓ (Figure 5). Assume that $b_1 = \min\{b_1, \ldots, b_n\}$, $b_n = \max\{b_1, \ldots, b_n\}$, and $m = \frac{b_n - b_1}{2}$.

2. Let ℓ' be a line parallel to ℓ and above ℓ such that the distance between ℓ and ℓ' is, for example, larger than $b_n - b_1$. Let $\{b'_n, \ldots, b'_1\}$ a vertical projection of the points $\{b_1, \ldots, b_n\}$ on ℓ' but in reverse order (Figure 5).

   Let ℓ_1 be the horizontal mid-line of the lines ℓ and ℓ'. Let m' be the vertical projection of m onto ℓ_1. Put two red points r_1 and r_2 in the positions $m' - \frac{\epsilon}{2}$ and $m' + \frac{\epsilon}{2}$ on ℓ_1, respectively.

   Compute the following parallel lines: line t_1 passing through b_1 and r_1 and line t_2 passing through $b_1 + \epsilon$ and r_2; and, similarly, line t_3 passing through b_n and r_2 and line t_4 passing through $b_n - \epsilon$ and r_1. Put a red point r_3 at the intersection between t_2 and t_4. Put a red point r_4 at the intersection between t_1 and t_3. Let R be the set of the four red points.

3. Put a blue point b_{n+1} (resp., b_{n+2}) in the mid point of the segment which endpoints are the intersection points of the line passing through r_1 (resp., r_2) and $b_1 - \frac{\epsilon}{2}$ (resp., $b_n + \frac{\epsilon}{2}$) with the two lines: the line ℓ_1 and the line ℓ_3 which is parallel to ℓ_1 and pass through r_3 (Figure 5). These two blue points avoid the existence of a separating strip other than the strip passing in between two consecutive blue points from $\{b_1, \ldots, b_n\}$. Analogously, but in a symmetric way, put the blue points b'_{n+1} and b'_{n+2} as it is shown in Figure 5 which are located in between the

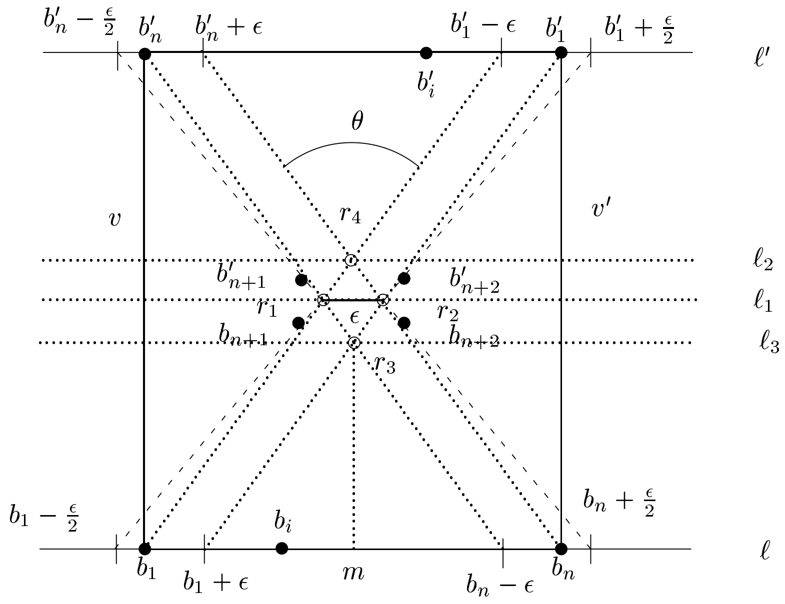

Figure 5: Strip through two given points.

lines ℓ_1 and ℓ_2, where ℓ_2 is the horizontal line passing through the red point r_4. Let B be the set of these $2n + 4$ blue points.

Now, any strip s that contains R and has boundary lines through r_1 and r_2 intersects both ℓ and ℓ' in intervals between v and v' having length ϵ, which should be free of points from B for s to be a separating strip. Therefore s separates R from B if and only if MAX-GAP$(S) \geq \epsilon$. □

Note that this strip separability only has one degree of freedom (the slope of the strip). Moreover, the slope interval of the possible strips can be previously determined by scaling the points $\{x_1, \ldots, x_n\}$ or by choosing an appropriate vertical distance between b_1 and b'_n.

3.2.3 Strip with fixed width

Now we consider the separability by a strip with a given width. Note that in Theorems 3 and 4 if R and B are strip separable, then they are separable by a strip of width w, which is at least $2a$ (Theorem 3) or at least a (Theorem 4). If d_r is the radius of the red circle, then the following formulas give the relation between the width w of the strip and d_r: $d_r^2 = (\epsilon/2)^2 + (w/2)^2$ in Theorem 3, $d_r^2 = (\epsilon/2)^2 + w^2$ in Theorem 4. Thus, we can reduce QGE to separability by a strip with fixed width w analogously as we did in Theorems 3 and 4, by using an appropriate choice of radius d_r for the red circle.

Corollary 1. *Deciding whether two disjoint point sets B and R are separable by a strip with a given width and such that one line of the strip passes through a given point has an $\Omega(n \log n)$ lower bound in the algebraic computation tree model.*

Remarks.

(1) If the strip passes through two given points and, moreover, we fix the width of the strip, then strip separability can be decided in linear time. Note that there are only two possible strips satisfying these conditions (Figure 6).

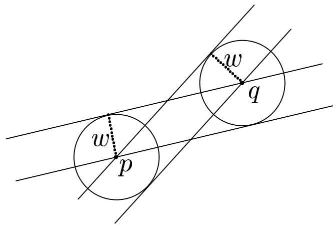

Figure 6: Strip separability fixing the width and two points.

(2) Fixing the slope of the strip, strip separability can be decided in linear time as follows. Assume that the given slope is the vertical slope. Project the points on a horizontal line, compute the projected red points with minimum and maximum x-coordinate, and check that there are no projected blue points in between these red points.

4 Wedge Separability

In [14], $O(n \log n)$ time algorithms are given for deciding wedge separability of point sets, as well as for constructing the locus of feasible apices. The wedges with maximum/minimum angle are also found. In this section we prove that those algorithms are worst-case optimal, by showing an $\Omega(n \log n)$ lower bound on deciding wedge separability.

4.1 Wedge separability in the general case

Simple geometric considerations show that due to the symmetry in the constructions in Theorems 3, 4 and 5 the sets R and B used there satisfy the following: if there exists a separating wedge of angle greater than 0 there is also a separating strip. On the other hand any separating strip is also a separating wedge (with angle 0). In other words, the sets R and B in those constructions are wedge separable if and only if they are strip separable. Hence the following theorem is immediately derived:

Theorem 6. *In the algebraic computation tree model, $\Omega(n \log n)$ operations are required for deciding whether two disjoint point sets are wedge separable in the general case, and the same lower bound applies when one of the rays bounding the wedge is constrained to contain a given point or even when both rays are constrained to contain to given points.*

4.2 Wedge with apex on a given line

Now we study separability by a wedge with the apex lying on a given line ℓ. If there are points of both sets above and below the line ℓ, then the existence of a separating wedge with apex on ℓ can be decided in $O(n)$ time as follows.

1. Compute the set B_1 (resp., B_2) of blue points above (resp., below) ℓ. (Blue points on ℓ are considered to be in both sets.)
2. Compute the interior supporting lines between R and B_1 and between R and B_2, using a prune and search algorithm described in [12] (without computing the convex hulls), if the respective pairs of sets are line separable. Refer to Figure 7, where the convex hulls are shown only for illustration. Determine the two intervals on ℓ given by the pairs of supporting lines, and

compute the intersection of these intervals. There exists a separating wedge with apex on ℓ if and only if the intersection is not empty.

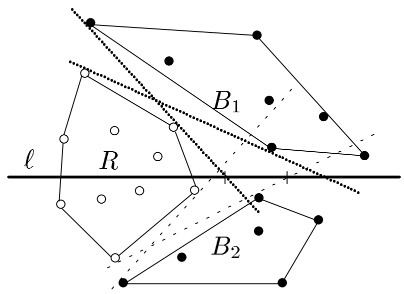

Figure 7: Case: Line ℓ crosses $CH(R)$.

Thus, we assume now (without loss of generality) that the red points are all above line ℓ. It is clear that the blue points that are below ℓ are irrelevant.

Theorem 7. *Deciding whether two disjoint point sets are separable by a wedge with apex on a given line ℓ requires $\Omega(n \log n)$ operations in the algebraic computation tree model.*

Proof. We give a linear-time reduction of Connected-Union (CU) to separability by a wedge with apex on line ℓ. Without loss of generality, suppose that ℓ is the x-axis. Let $\{x_1, \ldots, x_n\}$ and $\epsilon \in \mathbb{R}^+$ be an instance of CU. We have to decide whether the union of the intervals $(x_1 - \frac{\epsilon}{2}, x_1 + \frac{\epsilon}{2}), \ldots, (x_n - \frac{\epsilon}{2}, x_n + \frac{\epsilon}{2})$ is connected. We compute the minimum and the maximum of $\{x_1, \ldots, x_n\}$. Assume that x_1 is the minimum, x_n is the maximum, $a = x_1 - \frac{\epsilon}{2}$, and $b = x_n + \frac{\epsilon}{2}$. We do the following construction.

1. Represent the intervals $(x_1 - \frac{\epsilon}{2}, x_1 + \frac{\epsilon}{2}), \ldots, (x_n - \frac{\epsilon}{2}, x_n + \frac{\epsilon}{2})$ on ℓ.
2. Make a square of side $b - a$, as in Figure 8, and put red points, r_1 and r_2, on the top vertices of the square and a red point r_3 on the center of the square. Let R be the set of these three red points.

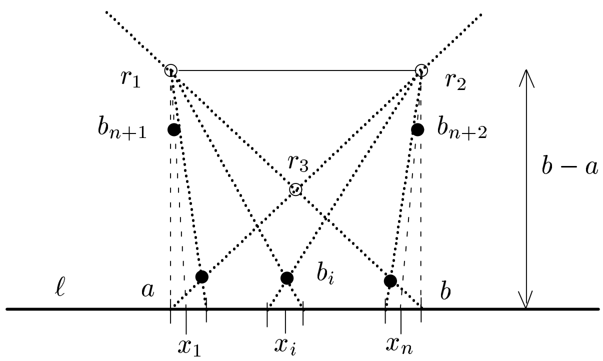

Figure 8: Wedge with apex on a given line: construction for proof of Theorem 7.

3. For each interval $(x_i - \frac{\epsilon}{2}, x_i + \frac{\epsilon}{2})$, compute the line passing through $x_i + \frac{\epsilon}{2}$ and r_1 and the line passing through $x_i - \frac{\epsilon}{2}$ and r_2; put a blue point b_i at the intersection point of these lines. Put

a blue point b_{n+1} (b_{n+2}) at the intersection point of the line $y = \frac{3(b-a)}{4}$ and the line passing through x_1 and r_1 (x_n and r_2) (See Figure 8). Let B be the set of these $n+2$ blue points.

Now, the points b_{n+1} and b_{n+2} prevent the existence of any wedge separating R from B, having apex on $(-\infty, a] \cup [b, +\infty)$. Similarly every point b_i prevents the apex of such a wedge to lie in the interval $(x_i - \frac{\epsilon}{2}, x_i + \frac{\epsilon}{2})$. Therefore, there exists a wedge with apex on ℓ separating R from B if and only if the union of the intervals $(x_1 - \frac{\epsilon}{2}, x_1 + \frac{\epsilon}{2}), \ldots, (x_n - \frac{\epsilon}{2}, x_n + \frac{\epsilon}{2})$ is not connected. Thus, Theorem 2 completes the proof. □

Note that we always can scale the numbers $\{x_1, \ldots, x_n\}$ and ϵ to lie on a given segment of the real line and therefore, same proof is valid for separability by a wedge with apex on a given segment.

4.2.1 Wedge with apex on a given line and a half-line with a given slope

If we know that the apex of the wedge lies on a given line (or segment) and one of the half-lines has a given slope, then the problem of the existence of a separating wedge still has an $\Omega(n \log n)$ lower bound by the following theorem.

Theorem 8. *Deciding whether two disjoint point sets are separable by a wedge such that the apex lies on a given line (or segment) and one of the half-lines has a given slope requires $\Omega(n \log n)$ operations in the algebraic computation tree model.*

Proof. In linear time, we reduce CU to separability by a wedge such that the apex lies on a given line (or segment) and one of the half-lines has a given slope. Suppose that the given line ℓ is the x-axis and that the given slope is the vertical slope. Let $\{x_1, \ldots, x_n\}$ and $\epsilon \in \mathbb{R}^+$ be an instance of CU. We have to decide whether the union of the intervals $(x_1 - \frac{\epsilon}{2}, x_1 + \frac{\epsilon}{2}), \ldots, (x_n - \frac{\epsilon}{2}, x_n + \frac{\epsilon}{2})$ is connected. We compute the minimum and the maximum of $\{x_1, \ldots, x_n\}$. Assume that x_1 is the minimum, x_n is the maximum, $a = x_1 - \frac{\epsilon}{2}$, and $b = x_n + \frac{\epsilon}{2}$. We do the following construction.

1. Represent the intervals $(x_1 - \frac{\epsilon}{2}, x_1 + \frac{\epsilon}{2}), \ldots, (x_n - \frac{\epsilon}{2}, x_n + \frac{\epsilon}{2})$ on ℓ.
2. Put two red points, r_1 and r_2, on a line parallel to ℓ such that the x-coordinates of r_1 and r_2 are larger than the x-coordinate of b (Figure 9). Let R be the set of these red points.

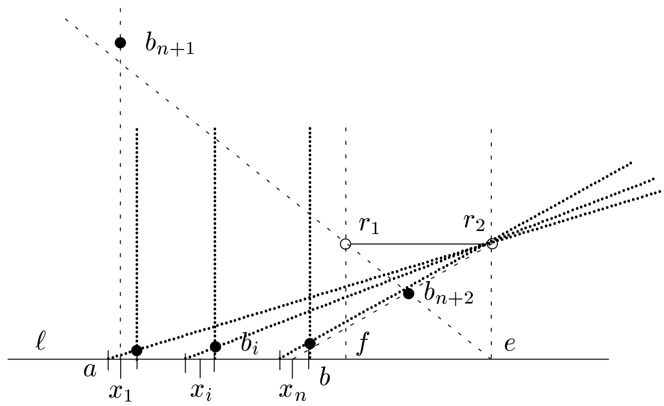

Figure 9: Apex on a given line and a vertical half-line.

3. For each interval $(x_i - \frac{\epsilon}{2}, x_i + \frac{\epsilon}{2})$, compute the line passing through $x_i - \frac{\epsilon}{2}$ and r_2 and the vertical line passing through $x_i + \frac{\epsilon}{2}$; put a blue point, b_i, at the intersection point of these lines. Put a blue point b_{n+1} on the vertical line passing through x_1 and such that its y-coordinate is larger than the y-coordinate of the intersection point between the vertical line passing through x_1 and the line passing through e and r_1 (Figure 9). Point b_{n+1} ensures that there are no separating wedges with apices in the intervals $(-\infty, a]$ and $[e, +\infty)$. Obviously, there are no separating wedges with apices in the interval (f, e). Put a blue point b_{n+2} at the intersection points of the lines: line passing through e and r_1 and line passing through x_n and r_2. Point b_{n+2} ensures that there are no separating wedges with apices in the interval $[b, f]$. Let B be the set of these $n + 2$ blue points.

Now, by construction, there exists a wedge separating R from B with apex on ℓ and one of the half-lines with vertical slope if and only if the union of the intervals $(x_1 - \frac{\epsilon}{2}, x_1 + \frac{\epsilon}{2}), \ldots, (x_n - \frac{\epsilon}{2}, x_n + \frac{\epsilon}{2})$ is not connected. Thus, Theorem 2 completes the proof. □

Remarks.

(1) If we know the line ℓ that contains one of the half-lines of the wedge (and therefore, ℓ contains the apex of the wedge), then wedge separability can be decided in linear time as follows. Assume that there are no points of both colors above and below the line ℓ; otherwise, there is no solution. Suppose that, for example, all the red points are above ℓ. In linear time we compute the set B_1 of blue points that are above ℓ and check linear separability between B_1 and R (see Figure 10).

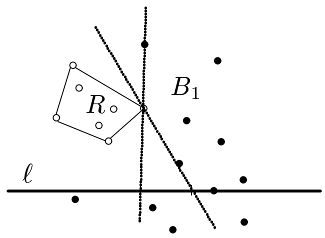

Figure 10: Wedge separability when it is known that one half-line lies on a given line ℓ.

(2) If we know the slopes of the half-lines of the wedge or, more generally, we know the slope of a half-line and a slope interval for the other half-line, then wedge separability can be decided in linear time as follows.

(a) Assume that the given slope of a half-line is the vertical slope given by a directed vertical line ℓ_1, and the red points are on the right side of ℓ_1. Let the other half-line be contained in a line ℓ_2, whose slope is within a given slope interval inside $[0, \frac{\pi}{2}]$ (see Figure 11). (Other cases are similar.)

(b) If there exists a separating wedge with the conditions above, then there exists also a separating wedge with a vertical half-line contained in ℓ'_1, the vertical directed line tangent to $CH(R)$, such that $CH(R)$ is on its right side (see Figure 11). Now, ℓ'_1 can be determined by computing the first red point on the right side of ℓ_1 as we sweep from left to right with a vertical line.

(c) Let B_1 be the set of blue points on the right side of ℓ'_1. Using linear programming, compute (in linear time) a line (if it exists) separating R from B_1 such that its slope is inside the given slope interval.

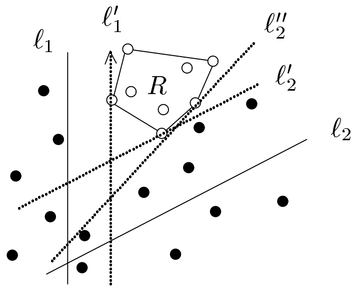

Figure 11: Wedge separability when the slopes of the half-lines are known.

(3) If we know the apex p of the wedge, deciding whether the sets B and R are wedge separable can be done in linear time as follows. Using linear programming, check that p is line separable from R. Compute the supporting lines from p to $CH(R)$ by, for example, computing the slopes of the lines passing through p and each red point and determining the slope extremes. Check that the wedge defined by the rays from p contained in the above supporting lines does not contain blue points.

5 Triangle Separability

Edelsbrunner and Preparata [7] have shown two algorithms for constructing a minimum-edge convex polygon that separates two sets of n points in the plane, if such a separator exists. The first algorithm takes $O(n \log n)$ time, and this is optimal in the worst case if the number of edges, k, is $k = \Theta(n)$. The second algorithm takes $O(kn)$ time for constructing a separating convex k-gon, where k is either optimal or one larger. The authors raised the following open problem:

Is there an $\Omega(n \log n)$ time lower bound for the construction of a separating convex k-gon for smallest k, even if k is small? More specifically, is $\Omega(n \log n)$ time required to decide whether or not there exists a separating triangle?

We give an affirmative answer to the latter question. Observe that the case $k = 1$ is that of linear separability, which can be decided in $\Theta(n)$ time, and the case $k = 2$ corresponds to wedge separability, which has an $\Omega(n \log n)$ lower bound as we have shown in Section 4.

5.1 Triangle separability in the general case

Theorem 9. *Deciding whether two disjoint point sets B and R are separable with a triangle requires $\Omega(n \log n)$ operations in the algebraic computation tree model.*

Proof. We show that QGE is linear-time reducible to triangle separability for two sets of points. Let $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ and $\epsilon \in \mathbb{R}^+$ be an instance of QGE. We have to decide whether the Euclidean distance between some two consecutive points in S is $\geq \epsilon$. We assume that S has at least three different points, otherwise the decision is trivial. We do the following.

1. If $\epsilon \geq \sqrt{2 - \sqrt{2}}$, then proceed as in steps 1 and 2 of Theorem 4.

2. If $\epsilon < \sqrt{2 - \sqrt{2}}$, do the following construction.

   (a) Consider the n input points to be the red points $(r_{x1}, r_{y1}), \ldots, (r_{xn}, r_{yn})$ in the first quadrant on the unit circle (the *red circle*). Compute the red points in the first quadrant with minimum and maximum x-coordinate; let them be denoted r_1 and r_n, respectively.

Compute the red point with the second smallest x-coordinate (call it r_2) and with the second largest x-coordinate (call it r_{n-1}). If the distance between x_1 and x_2 is $\geq \epsilon$ or the distance between x_{n-1} and x_n is $\geq \epsilon$ then the decision for QGE is affirmative and we are done. Thus, assume that the distance between x_1 and x_2 is $< \epsilon$ and the distance between x_{n-1} and x_n is $< \epsilon$. Place a blue point, b, at the intersection of the line passing through r_1 and r_2 and the line passing through r_{n-1} and r_n. Refer to Figure 12b.

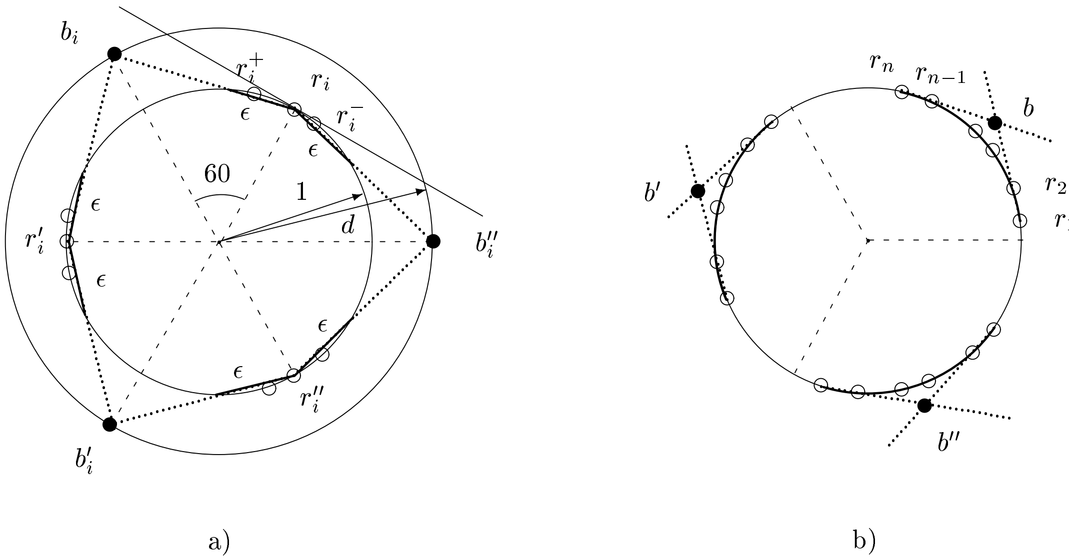

Figure 12: Triangle separability in the general case: construction for the proof of Theorem 9.

(b) Make a copy of the red points on the red circle and the point b, and rotate all $n+1$ points about the center by angle 120 degrees; let the resulting points be denoted $r'_1, r'_2, \ldots, r'_n$ and b' (i.e., for a point $r = (p, q)$ we produce a copy $r' = ((-p + \sqrt{3}q)/2, (\sqrt{3}p + q)/2))$. Similarly, make a second copy by rotating 240 degrees the red points and b, obtaining points $r''_1, r''_2, \ldots, r''_n$ and b''. Refer to Figure 12b.

(c) Consider a concentric circle (the *blue circle*) with radius

$$d = \frac{2\sqrt{4 - \epsilon^2}}{\epsilon\sqrt{3} + \sqrt{4 - \epsilon^2}}.$$

The reasons for the choice of d appear next; on the other hand it is easy to see that

$$2 \geq d \geq \frac{\sqrt{6} - 2 - \sqrt{2}}{2 - \sqrt{2}} \approx 1.16452 \quad \text{when } 0 \leq \epsilon \leq \sqrt{2 - \sqrt{2}}$$

Make a *rotated copy* of the red points $r_1, \ldots, r_n$ onto the blue circle as follows: every r_i is rotated 60 degrees around the origin and then projected from the origin in the blue circle in a point b_i (see Figure 12a). The value of d has been chosen in such a way that the segment $b_i r_i$ crosses the red circle in a point at distance exactly ϵ from r_i. The same construction is carried out for obtaining rotated copies on the blue circle of the red points $r'_1, \ldots, r'_n$ and $r''_1, \ldots, r''_n$. Notice that by the rotational symmetry the segment $b''_i r_i$ crosses the red circle in a point at distance exactly ϵ from r_i. Refer to Figure 12a.

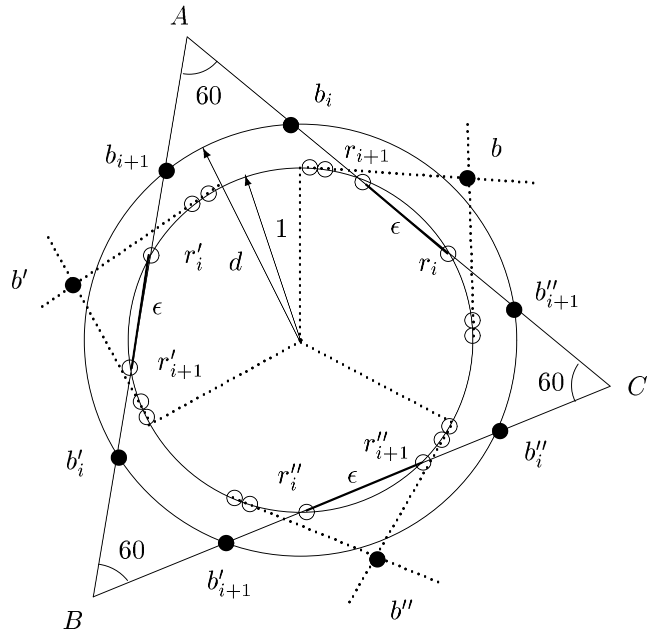

Figure 13: Triangle separability in the general case.

In this way we have obtained a set R of $3n$ red points and a set B of $3n+3$ blue points. We are going to prove that R and B are triangle separable if and only if the maximum gap determined by the points $r_1, \ldots, r_n$ is greater or equal than ϵ.

First notice that the role of the points b, b' and b'' consists of the fact that no two of them can be separated simultaneously from the red points by a single line (see Figure 12b). As a consequence any triangle T_1 separating R from B can be shrunk to a triangle T_2 with sides parallel to those of T_1 until one side l of T_2 contains a point r_i, another side l' contains a point r'_j, and the third side l'' contains a point r''_k. Furthermore, the line containing the side l separates b from $r_1, \ldots, r_n$, the line supporting l' separates b' from $r'_1, \ldots, r'_n$, and the line containing l'' separates b'' from $r''_1, \ldots, r''_n$. By the definition of the points b, b' and b'', we can assume that l contains a point r_i, $2 \leq i \leq n-1$, l' contains a point r'_j, $2 \leq j \leq n-1$ and l'' contains a point r_k, $2 \leq k \leq n-1$.

If the maximum gap determined by the points $r_1, \ldots, r_n$ is equal to ϵ and given by r_i and r_{i+1}, then the triangle ABC determined by lines $\ell = r_i b_i$, $\ell' = r'_i b'_i$ and $\ell'' = r''_i b''_i$ (refer to Figure 13) separates R from B because the triangle ABC intercepts on the blue circle the arcs $b_i b_{i+1}$, $b'_i b'_{i+1}$ and $b''_i b''_{i+1}$, whose interior is empty of blue points.

Notice that if the maximum gap determined by the points $r_1, \ldots, r_n$ is greater than ϵ and given by r_i and r_{i+1}, then still the triangle ABC with supporting lines $\ell = r_i b_i$, $\ell' = r'_i b'_i$ and $\ell'' = r''_i b''_i$ still separates R from B, as r_{i+1} moves toward the interior of ABC and b_{i+1} toward its exterior.

On the other hand, if the maximum gap determined by the points $R_1 = \{r_1, \ldots, r_n\}$ is smaller than ϵ, then for each r_i, $2 \leq i \leq n-1$, there are two red points r_i^- and r_i^+ in R_1 such that the distance between r_i and r_i^- is $< \epsilon$ and the distance between r_i and r_i^+ is $< \epsilon$. Assume that there exists a separating triangle T_2 and that the line supporting the side l contains the point r_i with $2 \leq i \leq n-1$. Due to the points r_i^- and r_i^+, the line containing l can not separate r_i from any of the points b_i and b''_i (see Figure 12a). Thus, the wedge defined by the lines containing the other two sides l' and l'' of T_2 has to separate the points b_i, b'_i and b''_i from the red points, but this is prevented by the points r'_i, $r_i'^+$, $r_i'^-$, r''_i, $r_i''^+$, $r_i''^-$. Hence we get a contradiction and see that no separating triangle can exists.

Therefore, we have proved that the sets of red and blue points are triangle separable if and only if the maximum gap of the red points in the first quadrant is $\geq \epsilon$. □

A related problem is to determine whether there exists a triangle separating two nested convex polygons (a red polygon contained inside a blue polygon). This is a different problem because we know the order of the edges of the polygons. In [1] (see also [20]), the authors give an $O(nk)$-time algorithm for finding a minimum-vertex polygon that separates two nested simple polygons, where n is the total number of vertices of the input polygons and k is the minimum number of vertices of a separator. This result yields an optimal $\Theta(n)$-time algorithm for the triangle separability of two nested convex polygons.

5.2 Triangle separability with constraints

Given some additional information about the location of the vertices of the triangle does not help to decide triangle separability, as we now show:

Theorem 10. *Deciding whether two disjoint point sets B and R are separable by a triangle with vertices lying on given lines requires $\Omega(n \log n)$ operations in the algebraic computation tree model.*

Proof. We reduce CU to separability by a triangle with vertices lying on given lines (in general position) by using the same construction we did in Theorem 7. By that construction, if there exists a separating triangle with a vertex on a line ℓ_1, this vertex has to lie in the interval (a, b) (see Figure 14); and, this is so if and only if the union of the intervals $(x_1 - \frac{\epsilon}{2}, x_1 + \frac{\epsilon}{2}), \ldots, (x_n - \frac{\epsilon}{2}, x_n + \frac{\epsilon}{2})$ is not connected.

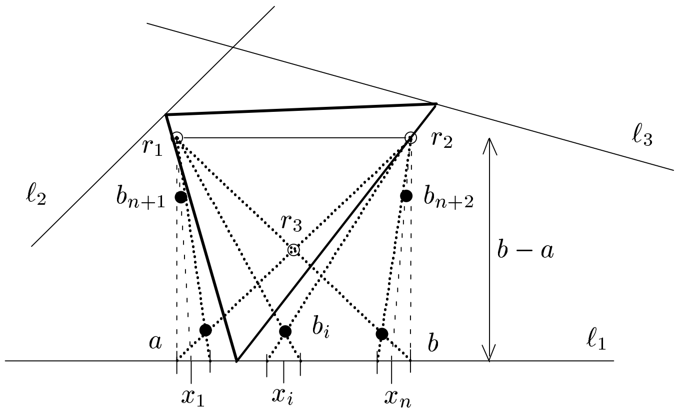

Figure 14: Triangle separability with vertices constrained to lie on given lines ℓ_1, ℓ_2, and ℓ_3.

Again, Theorem 2 completes the proof for us. Note that the intersection points of the half-lines of the separating wedge with the lines ℓ_2 and ℓ_3 determine the separating triangle (Figure 14). □

Remark. It can be seen that deciding the existence of a separating triangle is feasible in linear time when we are given one or two vertices of the triangle, or two lines supporting sides. A lower bound $\Omega(n \log n)$ holds for the case in which only a line containing one side is given; the construction is very similar to the preceding and omitted to avoid repetition. On the other hand, a lower bound $\Omega(n \log n)$ seems very likely when the sides are constrained to contain three given points; this would fit the best known algorithm but a proof remains elusive to us.

6 Fixed-Slopes Double Wedge Separability

Double wedge separability of the sets B and R can be decided in $O(n \log n)$ time, as proved in [15]. A particularly natural case to consider is that in which the slopes of the two lines are given (without loss of generality, say horizontal and vertical), as in Figure 15. In this case, it is reasonable to expect that a faster algorithm may be possible; however, we prove, using a reduction from the GE problem, that this is not the case.

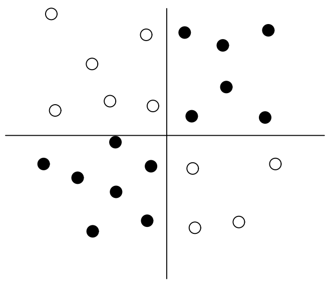

Figure 15: Vertical-horizontal double wedge separability.

Theorem 11. *Deciding whether two disjoint point sets B and R are separable by a double wedge with given slopes requires $\Omega(n \log n)$ operations in the algebraic computation tree model.*

Proof. First of all, we prove that the following problem has an $\Omega(n \log n)$ time lower bound.

Nontrivial Cross Split (CS) . *Given n points $\{a_1, \ldots, a_n\}$ in the plane, decide whether there exists a "cross" (vertical and horizontal line) that nontrivially splits the points, with at least one point in each of the two diagonally opposite quadrants (the other two quadrants are empty).*

Problem GE can be reduced to problem CS in linear time. Let $S = \{x_1, \ldots, x_n\}$ and $\epsilon \in \mathbb{R}^+$ be an instance of problem GE; we have to determine whether MAX-GAP(S) $\geq \epsilon$. We do the following construction.

1. Take each point x_i and create a point in the plane, $b_i = (x_i, x_i)$. This gives us a diagonal line of points $\{b_1, \cdots, b_n\}$.
2. Make a second copy and shift it by a distance $\epsilon\sqrt{2}$ to the northwest, obtaining the line of points $\{b'_1, \cdots, b'_n\}$ (see Figure 16).

Now, problem CS can be thought of as asking if the southeast "staircase" (locus of points for which the southeast quadrant is free of points of the input configuration) meets the northwest staircase. They meet if and only if there is a cross that separates. By the selection of the separation of two lines of points, this will happen if and only if MAX-GAP(S) $\geq \epsilon$. The reduction from problem GE to problem CS can be performed in linear time.

Moreover, we have also a reduction from problem CS to the double wedge separability problem. The reduction is as follows. The above input of problem CS is converted into the following double wedge separability input.

1. Set B is formed by the union of the above point sets, i.e., $B = \{b_1, \cdots, b_n, b'_1, \cdots, b'_n\}$, where $b_i = (x_{bi}, y_{bi})$ and $b'_i = (x_{b'i}, y_{b'i})$.

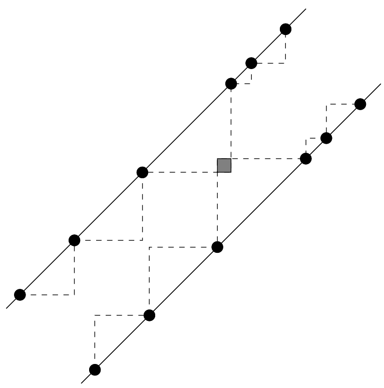

Figure 16: Construction for the lower bound for Nontrivial Cross Split.

2. Set R is formed by two points r_1, r_2, where $r_1 = (x_{r1}, y_{r1})$ and $r_2 = (x_{r2}, y_{r2})$ such that:

$$x_{r1} < \min\{x_{b1}, \cdots, x_{bn}, x_{b'1}, \cdots, x_{b'n}\}, \qquad y_{r1} > \max\{y_{b1}, \cdots, y_{bn}, y_{b'1}, \cdots, y_{b'n}\},$$

$$x_{r2} > \max\{x_{b1}, \cdots, x_{bn}, x_{b'1}, \cdots, x_{b'n}\}, \qquad y_{r2} < \min\{y_{b1}, \cdots, y_{bn}, y_{b'1}, \cdots, y_{b'n}\}.$$

Now, the sets B and R are double wedge separable if and only if there exists a nontrivial cross split of B. The reduction from problem CS to double wedge separability problem can be performed in linear time. The two reductions show the $\Omega(n \log n)$ lower bound for the double wedge separability with fixed-slopes. □

Remarks.

(1) The same proof is valid if we restrict the apex of the double wedge to lie on a given line ℓ. Take ℓ as the strip axis of the strip defined by the parallel lines that contain the points $\{b_1, \cdots, b_n\}$ and $\{b'_1, \cdots, b'_n\}$. In that case, we have only one degree of freedom, but still an $\Omega(n \log n)$ lower bound.

(2) If we know one line of the double wedge, then deciding double wedge separability can be done in linear time as follows.

(a) Let ℓ_1 be the given line. Suppose that ℓ_1 is a horizontal line. Let B_1 (resp., R_1) be the set of blue (resp., red) points above ℓ_1. Let B_2 (resp., R_2) be the set of blue (resp., red) points below ℓ_1. Then, B_1, R_1, B_2 and R_2 are not empty sets, since ℓ_1 has to separate some red and blue points (at least one of each color on each side of ℓ_1). Suppose, for the moment, that ℓ_1 does not contain blue and red points.

(b) In linear time, compute a line ℓ_2 separating $B_1 \cup R_1$ and $B_2 \cup R_2$. Lines ℓ_1 and ℓ_2 form a separating double wedge. Now, if there are red or blue points on ℓ_1, assign red points to R_1 or R_2 and blue points to B_1 or B_2, as suitable.

(3) If we know the apex p of the double wedge, we can decide in linear time whether B and R are double wedge separable. The key idea is to observe that if the points are separable by a double wedge with apex p, then there exists a vertical or horizontal line passing through p giving the same partition of one of the sets as the double wedge. Using this idea we proceed as follows.

(a) Assume that the vertical line passing through p gives the partition of the red points of the double wedge with apex p (proceed analogously in other cases). Compute this partition, R_1,

R_2. Compute the supporting lines from p to $CH(R_1)$ and the supporting lines from p to $CH(R_2)$, without computing the convex hulls (see Figure 17).

(b) Compute the partition, B_1 and B_2, produced by the supporting lines above and check that $B = B_1 \cup B_2$. Then, using linear programming, determine two lines passing through p, one separating $R_1 \cup B_1$ from $R_2 \cup B_2$, and the other one separating $R_1 \cup B_2$ from $R_2 \cup B_1$. The two lines form the separating double wedge.

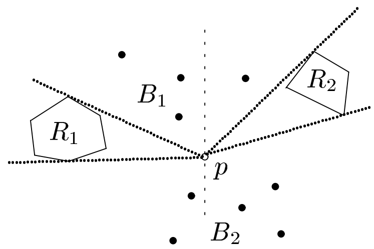

Figure 17: Double wedge with apex p.

7 Conclusion

When B and R are not strip separable, it is possible to consider a separating strip such that the number of misclassified points is minimized. An $O(n \log n)$ time algorithm for solving this problem is described in [2]. This minimum may be zero, in which case the sets are strip separable. Thus, the $\Omega(n \log n)$ lower bound for strip separability applies also to the problem of minimizing the number of misclassified points by a strip. The same observation applies to the problem of minimizing the number of misclassified points in wedge separability, for which an $O(n^2)$ time algorithm is described in [2].

Notice that the separability problems we have studied fall into two categories: $\Theta(n)$-time separability problems or $\Theta(n \log n)$-time separability problems; refer to Table 1 for a summary of our results. It would be interesting to determine more generally what properties of a particular problem instance distinguishes which of the two categories it falls into. Also, are there any separability problems for points in the plane whose complexity falls between $\Theta(n)$ and $\Theta(n \log n)$?

Finally it is worth nothing that beyond the triangle, other sets of three linear objects can used to separated points, e.g. three lines, a wedge and a line, a 3-path (sequences of ray-segment-ray), etc. To the best of our knowledge only the case of three lines has been studied [2, 6].

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References

[1] A. Aggarwal, H. Booth, J. O'Rourke, S. Suri, *Finding minimal convex nested polygons*, Information and Computation, 83, 1989, pp. 98–110.

[2] E. M. Arkin, F. Hurtado, J. S. B. Mitchell, C. Seara, S. S. Skiena, *Some separability problems in the plane*, 16th European Workshop on Computational Geometry, 2000, pp. 51–54.

Table 1: Summary of our results.

Criteria of separability: two parallel lines	Complexity
Strip	$\Theta(n \log n)$
Strip passing through a given point	$\Theta(n \log n)$
Strip passing through two given points	$\Theta(n \log n)$
Strip with given width	$\Theta(n \log n)$
Strip with given width and passing through two points	$\Theta(n)$
Strip with a given slope	$\Theta(n)$
Criteria of separability: two crossing lines	Complexity
Double wedge with fixed-slopes	$\Theta(n \log n)$
Double wedge with fixed-slope and apex on a given line	$\Theta(n \log n)$
Double wedge with a given line	$\Theta(n)$
Double wedge with a given apex	$\Theta(n)$
Criteria of separability: two rays with common origin	Complexity
Wedge	$\Theta(n \log n)$
Wedge with apex on a given line	$\Theta(n \log n)$
Wedge with apex on a given line and a half-line with a given slope	$\Theta(n \log n)$
Wedge with a half-line on a given line	$\Theta(n)$
Wedge with given slopes of the half-lines	$\Theta(n)$
Wedge with given apex	$\Theta(n)$
Criteria of separability: Three lines	Complexity
Triangle	$\Theta(n \log n)$
Triangle with vertices on given lines	$\Theta(n \log n)$
Triangle with one side on a given line	$\Theta(n \log n)$
Triangle with two sides on given lines	$\Theta(n)$
Triangle with one or two given vertices	$\Theta(n)$

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