Geodeticity of the contour of chordal graphs

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Abstract

A vertex v is a *boundary vertex* of a connected graph G if there exists a vertex u such that no neighbor of v is further away from u than v. Moreover, if no vertex in the whole graph $V(G)$ is further away from u than v, then v is called an *eccentric vertex* of G. A vertex v belongs to the *contour* of G if no neighbor of v has an eccentricity greater than the eccentricity of v. Furthermore, if no vertex in the whole graph $V(G)$ has an eccentricity greater than the eccentricity of v, then v is called a *peripheral vertex* of G. This paper is devoted to study these kinds of vertices for the family of chordal graphs. Our main contributions are, firstly, obtaining a realization theorem involving the cardinalities of the periphery, the contour, the eccentric subgraph and the boundary, and secondly, proving both that the contour of every chordal graph is geodetic and that this statement is not true for every perfect graph.

Key words: Boundary, contour, convex hull, convexity, geodetic set, chordal graph, perfect graph.

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1 Introduction

The extraordinary development during the last few decades of a number of discrete and combinatorial mathematical structures has lead to the creation and study of distinct analogies and generalizations of a number of classical concepts, ideas, and methods from continuous mathematics. Among them, the notion of convex set of a metric space and the convex hull operator play a significant role [12]. Since connected graphs can be seen as metric spaces just by considering their shortest paths, this fact has lead to the study of the behavior of these structures as convexity spaces [5,12,14].

All graphs in this paper are finite, undirected, simple and connected. For undefined basic concepts we refer the reader to introductory graph theoretical literature, e.g., [13]. Given vertices u, v in a graph $G = (V, E)$, the *geodetic interval* $I[u, v]$ is the set of vertices of all $u - v$ shortest paths. Given $W \subseteq V$, the *geodetic closure* $I[W]$ of W is the union of all geodetic intervals $I[u, v]$ over all pairs $u, v \in W$, i.e., $I[W] = \bigcup_{u,v \in W} I[u, v]$. A vertex set $W \subseteq V$ is called *convex* if $W = I[W]$. The smallest convex set containing W is denoted $[W]$ and is called the *convex hull* of W. Certainly, $W \subseteq I[W] \subseteq [W] \subseteq V$. A vertex set $W \subseteq V$ is called: a *hull set* if $[W] = V$ and a *geodetic set* if $I[W] = V$.

Given a graph $G = (V, E)$ and a convex set $W \subseteq V$, a vertex $v \in W$ is called an *extreme vertex* of W if the set $W \setminus \{v\}$ is also convex. The graph G is called a *convex geometry* (or also an *antimatroid*) if it satisfies the so-called *Krein-Milman* property: *Every convex set of G is the convex hull of its extreme vertices* [5,9,10]. Certainly, this condition can be seen as a procedure that may allow us to recover any convex set from its extreme vertices, by means of the convex hull operator. Under this point of view, the interest of any similar property is that a small subset of any convex set keeps all the information of the whole set.

A graph G is *chordal* if it contains no induced cycle of length greater than 3. A chordal graph is called *Ptolemaic* if it is distance-hereditary, that is, if every chordless path is a shortest path. Farber and Jamison [5] proved that a graph G is a convex geometry if and only if G is Ptolemaic. Thus, we could think of extending this property in two different ways. On the one hand, recovering convex sets on wider graph classes and, on the other hand, using an operator simpler than the convex hull one, such as for example the geodetic closure operator I. In both cases, finding new vertex sets playing a similar role to that of extreme vertices is necessary.

Concerning the first mentioned extension of the Krein-Milman property, Cáceres et al. [3] obtained a similar property valid for every graph, by considering, instead of the extreme vertices, the so-called *contour vertices.* As for the second generalization, consisting of using the geodetic closure operator, a number of results have recently been obtained [2,3,7,8]. For example, it has been proved that in the class of distance-hereditary graphs, every convex set is the geodetic closure of its contour vertices [3,8].

This work is mainly devoted to study several types of *boundary vertex* sets [4] for the family of chordal graphs. Our main contributions are, firstly, obtaining a realization theorem involving four of these sets (boundary, eccentric subgraph, contour and periphery), and secondly, proving both that the contour of every chordal graph is geodetic and that this statement is not true for every perfect graph.

2 Chordal graphs with specified boundary vertex counts

2.1 Boundary vertex sets

Let $G = (V, E)$ be a connected graph and $u, v \in V$. A vertex v is a *boundary vertex of* u if no neighbor of v is further away from u than v [4]. A vertex v is called a *boundary vertex* of G if it is the boundary vertex of some vertex $u \in V$. The *boundary* $\partial(G)$ of G is the set of all of its boundary vertices:

$$\partial(G) = \{v \in V \mid \exists u \in V \text{ s.t. } \forall w \in \mathrm{N}(v) : d(u, w) \leq d(u, v)\}.$$

Given $u, v \in V$, the vertex v is called an *eccentric vertex of* u if no vertex in V is further away from u than v, that is, if $d(u, v) = \mathrm{ecc}(u) = \max\{d(u, v) \mid v \in V\}$. A vertex v is called an *eccentric vertex* of G if it is the eccentric vertex of some vertex $u \in V$. The *eccentric subgraph* $\mathrm{Ecc}(G)$ of G is the set of all of its eccentric vertices [4]:

$$\mathrm{Ecc}(G) = \{v \in V \mid \exists u \in V \text{ s.t. } \mathrm{ecc}(u) = d(u, v)\}.$$

A vertex $v \in V$ is called a *contour vertex* of G if no neighbor vertex of v has an eccentricity greater than $\mathrm{ecc}(v)$.

Definition 1 [3] *The contour* $\mathrm{Ct}(G)$ *of* G *is the set all of its contour vertices:*

$$\mathrm{Ct}(G) = \{v \in V \mid \mathrm{ecc}(u) \leq \mathrm{ecc}(v), \forall u \in \mathrm{N}(v)\}.$$

A vertex $v \in V$ is called a *peripheral vertex* of G if no vertex in V has an eccentricity greater than $\mathrm{ecc}(v)$, that is, if the eccentricity of v is exactly equal to the diameter $D(G)$ of G. The *periphery* $\mathrm{Per}(G)$ of G is the set all of its peripheral vertices:

$$\mathrm{Per}(G) = \{v \in V \mid \mathrm{ecc}(u) \leq \mathrm{ecc}(v), \forall u \in V\} = \{v \in V \mid \mathrm{ecc}(v) = D(G)\}.$$

Notice that every extreme vertex is a contour vertex, i.e., $\mathrm{Ext}(G) = \{v \in V \mid V \setminus \{v\} \text{ is convex}\} \subseteq \mathrm{Ct}(G)$. It is also clear that: $\mathrm{Per}(G) \subseteq \mathrm{Ct}(G) \cap \mathrm{Ecc}(G)$ and $\mathrm{Ecc}(G) \cup \mathrm{Ct}(G) \subseteq \partial(G)$.

2.2 On the sizes of boundary vertex sets

The cardinalities of the boundary vertex sets of any graph G must satisfy certain nontrivial constraints, specified by Theorem 2, proved in [2].

Theorem 2 [2] *Let $G = (V, E)$ be a connected graph.*

(1) If $|\operatorname{Per}(G)| = |\operatorname{Ct}(G)| = 2$, then either $|\partial(G)| = 2$ or $|\partial(G)| \geq 4$.
(2) If $|\operatorname{Ecc}(G)| = |\operatorname{Per}(G)| + 1$, then $|\partial(G)| > |\operatorname{Ecc}(G)|$.

As a direct consequence of these properties, the following result is immediately derived.

Lemma 3 [2] *Let G be a nontrivial connected graph such that $|\operatorname{Per}(G)| = a$, $|\operatorname{Ct}(G)| = b$, $|\operatorname{Ecc}(G)| = c$ and $|\partial(G)| = d$. Then,*

$$\begin{cases} 2 \leq a \leq \min\{b, c\} \leq \max\{b, c\} \leq d, \\ (a, b, c, d) \neq (2, 2, 2, 3), \\ (a, b, c, d) \neq (a, b, a + 1, a + 1). \end{cases} \tag{1}$$

In the same paper [2], the following realization theorem was proved.

Theorem 4 [2] *Let $(a, b, c, d) \in \mathbb{Z}^4$ be integers satisfying the constraints (1) of Lemma 3. Then, there exists a connected graph $G = (V, E)$ such that $|\operatorname{Per}(G)| = a$, $|\operatorname{Ct}(G)| = b$, $|\operatorname{Ecc}(G)| = c$, and $|\partial(G)| = d$.*

2.3 Existence of chordal graphs with prescribed boundary vertex counts

At this point, we restrict ourselves to the class of chordal graphs and we pose the following question: *If G is a chordal graph, are there further restrictions concerning the cardinalities of the sets $\operatorname{Per}(G)$, $\operatorname{Ct}(G)$, $\operatorname{Ecc}(G)$ and $\partial(G)$?* Next, we present a realization theorem showing the answer to be negative. We first give out a technical lemma, whose proof is straightforward and hence omitted.

Lemma 5 *Let $G = (V, E)$ be a connected nontrivial graph, $x \in V$ and $\lambda \geq 1$. Let $\widehat{G}$ be the graph obtained from G by replacing the vertex x by a complete graph K_λ and joining every vertex of K_λ to every neighbor of x in G. Let $\eta : \widehat{G} \to G$ be the homomorphism that corresponds to contraction of that K_λ to x, Then,*

(1) $\operatorname{ecc}_{\widehat{G}}(v) = \operatorname{ecc}_G(\eta(v))$ for all $v \in V(\widehat{G})$.
(2) If $x \in \operatorname{Per}(G)$, then $\operatorname{Per}(\widehat{G}) = \operatorname{Per}(G) \cup V(K_\lambda)$.

(3) If $x \in \mathrm{Ct}(G)$, then $\mathrm{Ct}(\widehat{G}) = \mathrm{Ct}(G) \cup V(K_\lambda)$.
(4) If $x \in \mathrm{Ecc}(G)$, then $\mathrm{Ecc}(\widehat{G}) = \mathrm{Ecc}(G) \cup V(K_\lambda)$.
(5) If $x \in \partial(G)$, then $\partial(\widehat{G}) = \partial(G) \cup V(K_\lambda)$.
(6) $\widehat{G}$ is chordal if and only if G is chordal.

Theorem 6 *Let $(a, b, c, d) \in \mathbb{Z}^4$ be integers satisfying the constraints (1) of Lemma 3. Then, there exists a connected chordal graph $G = (V, E)$ such that*

$$|\operatorname{Per}(G)| = a, \quad |\operatorname{Ct}(G)| = b, \quad |\operatorname{Ecc}(G)| = c, \quad |\partial(G)| = d.$$

PROOF. Let's partition the set of all valid quadruples (a, b, c, d) into the twelve cases shown in Table 1.

Table 1: Possible cases in the proof of theorem 6.

(1)	$2 \le a = b = c = d$
(2)	$2 \le a < c < b < d$
(3)	$2 \le a < b < c < d$
(4)	$2 \le a = b < c < d$
(5)	$2 \le a < b = c < d$
(6)	$2 \le a < b < c = d$
(7)	$2 \le a = c < b < d$
(8)	$2 \le a < c < b = d$
(9)	$2 \le a < b = c = d$ s. t. $(a, b, c, d) \neq (a, b, a+1, a+1)$
(10)	$2 \le a = b < c = d$ s. t. $(a, b, c, d) \neq (a, b, a+1, a+1)$
(11)	$2 \le a = b = c < d$ s. t. $(a, b, c, d) \neq (2, 2, 2, 3)$
(12)	$2 \le a = c < b = d$

Clearly, the complete graph K_a satisfies the desired properties for case (1). For the remaining cases, we take a fixed *base graph* G, specific to each case, and then replace selected vertices by sets of twin vertices, as in Lemma 5, so as to expand the sets ∂, Ecc, Ct, Per to the required cardinalities. Since all base graphs are chordal, and the expansion η^{-1} preserves chordality, the expanded graph $\widehat{G}$ satisfies the theorem. More precisely, the proof is based on the following construction:

(1) Let G be the chordal graph shown on Figure 1 appropriate to the case.
(2) Replace a vertex $v_1 \in \operatorname{Per}(G)$ by the complete graph K_{a-h}, where $h = |\operatorname{Per}(G)| - 1$ (Lemma 5).
(3) If $a < b \le c$, replace a vertex $v_2 \in \operatorname{Ct}(G) \smallsetminus \operatorname{Per}(G)$ by the complete graph K_{b-a-h}, where $h = |\operatorname{Ct}(G)| - |\operatorname{Per}(G)| - 1$.
    If $a < c < b$, replace a vertex $v_2 \in \operatorname{Ecc}(G) \smallsetminus \operatorname{Per}(G)$ by the complete graph K_{c-a-h}, where $h = |\operatorname{Ecc}(G)| - |\operatorname{Per}(G)| - 1$.
(4) If $b < c$, replace a vertex $v_3 \in \operatorname{Ecc}(G) \smallsetminus \operatorname{Ct}(G)$ by the complete graph K_{c-b-h}, where $h = |\operatorname{Ecc}(G)| - |\operatorname{Ct}(G)| - 1$.
    If $c < b$, replace a vertex $v_3 \in \operatorname{Ct}(G) \smallsetminus \operatorname{Ecc}(G)$ by the complete graph K_{b-c-h}, where $h = |\operatorname{Ct}(G)| - |\operatorname{Ecc}(G)| - 1$.

(5) If $b \leq c < d$, replace a vertex $v_4 \in \partial(G) \smallsetminus \mathrm{Ecc}(G)$ by the complete graph K_{d-c-h}, where $h = |\partial(G)| - |\,\mathrm{Ecc}(G)| - 1$.

If $c < b < d$, replace a vertex $v_4 \in \partial(G) \smallsetminus \mathrm{Ct}(G)$ by the complete graph K_{d-b-h}, where $h = |\partial(G)| - |\,\mathrm{Ct}(G)| - 1$.

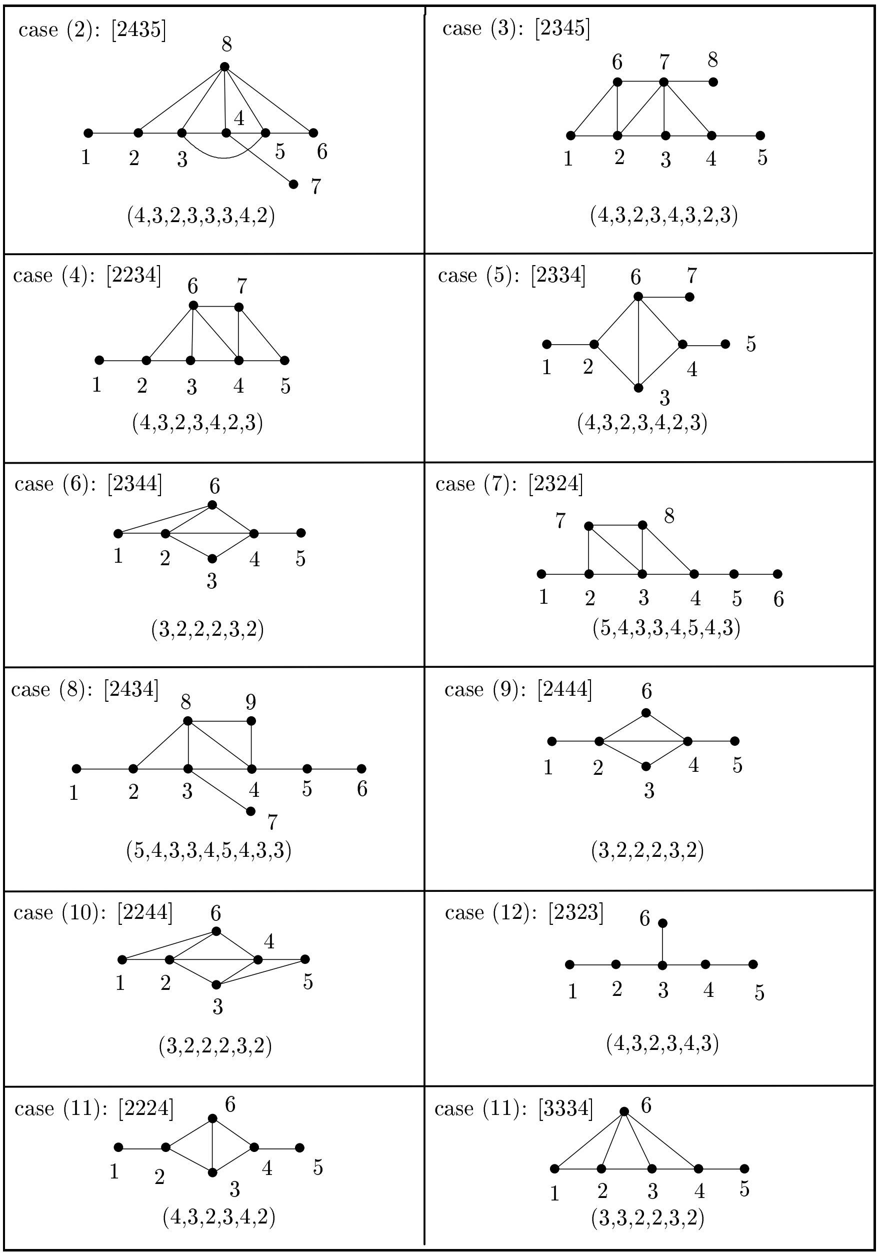

Figure 1. The base graphs for each case in the proof of Theorem 6. The vector below each graph is its eccentricity sequence.

Notice that, in all the graphs in Figure 1, either $\mathrm{Ct}(G) \subseteq \mathrm{Ecc}(G)$, or $\mathrm{Ecc}(G) \subseteq \mathrm{Ct}(G)$ (see Table 2).

Table 2: Boundary vertex sets of the base graphs.

G	$\mathrm{Per}(G)$	$\mathrm{Ct}(G)$	$\mathrm{Ecc}(G)$	$\partial(G)$	case
[2435]	$\{1,7\}$	$\{1,5,6,7\}$	$\{1,6,7\}$	$\{1,3,5,6,7\}$	(2)
[2345]	$\{1,5\}$	$\{1,5,8\}$	$\{1,5,6,8\}$	$\{1,3,5,6,8\}$	(3)
[2234]	$\{1,5\}$	$\{1,5\}$	$\{1,5,7\}$	$\{1,3,5,7\}$	(4)
[2334]	$\{1,5\}$	$\{1,5,7\}$	$\{1,5,7\}$	$\{1,3,5,7\}$	(5)
[2344]	$\{1,5\}$	$\{1,3,5\}$	$\{1,3,5,6\}$	$\{1,3,5,6\}$	(6)
[2324]	$\{1,6\}$	$\{1,6,7\}$	$\{1,6\}$	$\{1,6,7,8\}$	(7)
[2434]	$\{1,6\}$	$\{1,6,7,9\}$	$\{1,6,7\}$	$\{1,6,7,9\}$	(8)
[2444]	$\{1,5\}$	$\{1,3,5,6\}$	$\{1,3,5,6\}$	$\{1,3,5,6\}$	(9)
[2244]	$\{1,5\}$	$\{1,5\}$	$\{1,3,5,6\}$	$\{1,3,5,6\}$	(10)
[2224]	$\{1,5\}$	$\{1,5\}$	$\{1,5\}$	$\{1,3,5,6\}$	(11)
[3334]	$\{1,2,5\}$	$\{1,2,5\}$	$\{1,2,5\}$	$\{1,2,3,5\}$	(11)
[2323]	$\{1,5\}$	$\{1,5,6\}$	$\{1,5\}$	$\{1,5,6\}$	(12)

For the sake of clarity, we show the above construction for two cases.

Case (2)

(1) Take the graph $G = [2435]$ (Figure 1), which satisfies: $\mathrm{Per}(G) = \{1,7\}$, $\mathrm{Ct}(G) = \{1,5,6,7\}$, $\mathrm{Ecc}(G) = \{1,6,7\}$ and $\partial(G) = \{1,3,5,6,7\}$ (see Table 2).
(2) Replace vertex $1 \in \mathrm{Per}(G)$ by K_{a-1}.
(3) Replace vertex $6 \in \mathrm{Ecc}(G) \setminus \mathrm{Per}(G)$ by K_{c-a}.
(4) Replace vertex $5 \in \mathrm{Ct}(G) \setminus \mathrm{Ecc}(G)$ by K_{b-c}.
(5) Replace vertex $3 \in \partial(G) \setminus \mathrm{Ct}(G)$ by K_{d-b}.

For example, the graph obtained for $(a,b,c,d) = (4,8,6,10)$ is showed in Figure 2.

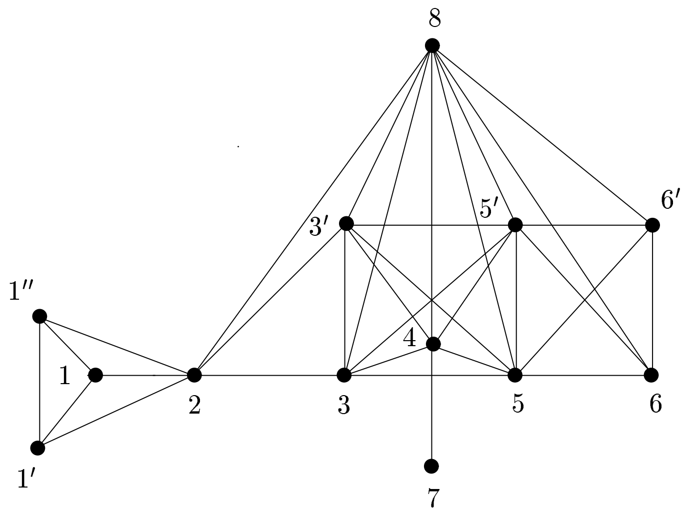

Figure 2. Graph G satisfying $|\,\mathrm{Per}(G)| = 4$, $|\,\mathrm{Ct}(G)| = 8$, $|\,\mathrm{Ecc}(G)| = 6$, and $|\partial(G)| = 10$.

Case (11). We must distinguish two subcases:

1. $a = 2$.
   (1) Take the graph $G = [2224]$ (Figure 1), which satisfies: $\mathrm{Per}(G) = \{1, 5\}$, $\mathrm{Ct}(G) = \{1, 5\}$, $\mathrm{Ecc}(G) = \{1, 5\}$ and $\partial(G) = \{1, 3, 5, 6\}$ (see Table 2).
   (5) Replace vertex $3 \in \partial(G) \setminus \mathrm{Per}(G)$ by K_{d-3}.
2. $a \geq 3$.
   (1) Take the graph $G = [3334]$ (Figure 1), which satisfies: $\mathrm{Per}(G) = \{1, 2, 5\}$, $\mathrm{Ct}(G) = \{1, 2, 5\}$, $\mathrm{Ecc}(G) = \{1, 2, 5\}$ and $\partial(G) = \{1, 2, 3, 5\}$ (see Table 2).
   (2) Replace vertex $1 \in \mathrm{Per}(G)$ by K_{a-2}.
   (5) Replace vertex $3 \in \partial(G) \setminus \mathrm{Per}(G)$ by K_{d-a}.

2.4 Existence of chordal graphs with specified extreme vertex counts

We will now prove a variant of Theorem 6, where the contour $\mathrm{Ct}(G)$ is replaced by the extreme set $\mathrm{Ext}(G)$. Firstly, notice that for nine of the twelve chordal graphs displayed in Figure 1, we have $\mathrm{Ct}(G) = \mathrm{Ext}(G)$. To be more precise, the only graphs for which $\mathrm{Ext}(G) \subsetneq \mathrm{Ct}(G)$ are [2435], [2324], and [3334]. In Figure 3, three new base chordal graphs are illustrated, having the same boundary parameters than the previous ones and satisfying the additional property $\mathrm{Ct}(G) = \mathrm{Ext}(G)$, as showed in Table 3.

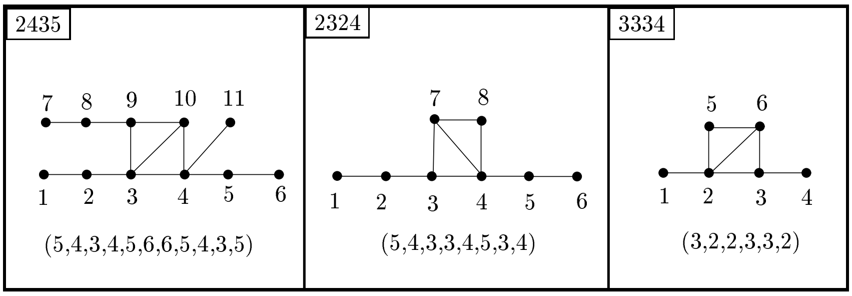

Figure 3. Some chordal graphs satisfying $\mathrm{Ext}(G) = \mathrm{Ct}(G)$.

Table 3: $\mathrm{Ct}(G) = \mathrm{Ext}(G)$.

G	$\mathrm{Per}(G)$	$\mathrm{Ct}(G) = \mathrm{Ext}(G)$	$\mathrm{Ecc}(G)$	$\partial(G)$
[2435]	$\{6, 7\}$	$\{1, 6, 7, 11\}$	$\{1, 6, 7\}$	$\{1, 6, 7, 10, 11\}$
[2324]	$\{1, 6\}$	$\{1, 6, 8\}$	$\{1, 6\}$	$\{1, 6, 7, 8\}$
[3334]	$\{1, 4, 5\}$	$\{1, 4, 5\}$	$\{1, 4, 5\}$	$\{1, 4, 5, 6\}$

It is well known that every nontrivial chordal graph contains at least two extreme vertices. These facts allow us to immediately derive the following realization theorem.

Theorem 7 *Let* $(a, b, c, d) \in \mathbb{Z}^4$ *be integers satisfying the constraints* (1) *of Lemma 3. Then, there exists a connected chordal graph* G *such that*

$$|\operatorname{Per}(G)| = a, \quad |\operatorname{Ext}(G)| = b, \quad |\operatorname{Ecc}(G)| = c, \quad |\partial(G)| = d.$$

Observe that, contrarily to Theorem 6, this result does not completely resolve the posed problem since the constraint $2 \leq a \leq \min\{b, c\} \leq \max\{b, c\} \leq d$ is not necessary when considering the extreme set instead of the contour. For example, the graph [3334] in Figure 1 satisfies $|\operatorname{Ext}(G)| = 2$ and $|\operatorname{Per}(G)| = 3$.

2.5 Containment of eccentric and contour vertices

Finally, observe that all the chordal graphs displayed in Figure 1 satisfy either $\operatorname{Ct}(G) \subseteq \operatorname{Ecc}(G)$ or $\operatorname{Ecc}(G) \subseteq \operatorname{Ct}(G)$. As a matter of fact, this condition has been decisive to prove both realization theorems. Nevertheless, this property is far for being true for every chordal graph. For example, each of the graphs showed in Figure 4 verifies neither of the above inclusions (see Table 4).

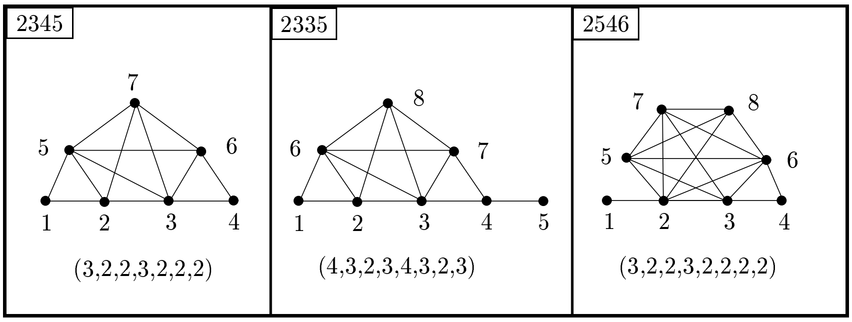

Figure 4. Some chordal graphs satisfying $\operatorname{Ct}(G) \neq \operatorname{Ct}(G) \cup \operatorname{Ecc}(G) \neq \operatorname{Ecc}(G)$.

Table 4: Neither $\operatorname{Ct}(G) \subseteq \operatorname{Ecc}(G)$ nor $\operatorname{Ecc}(G) \subseteq \operatorname{Ct}(G)$.

G	$\operatorname{Per}(G)$	$\operatorname{Ct}(G)$	$\operatorname{Ecc}(G)$	$\partial(G)$
[2345]	$\{1, 4\}$	$\{1, 4, 7\}$	$\{1, 2, 4, 6\}$	$\{1, 2, 4, 6, 7\}$
[2335]	$\{1, 5\}$	$\{1, 5, 8\}$	$\{1, 2, 5\}$	$\{1, 2, 5, 7, 8\}$
[2546]	$\{1, 4\}$	$\{1, 4, 5, 7, 8\}$	$\{1, 3, 4, 8\}$	$\{1, 3, 4, 5, 7, 8\}$

3 Geodeticity of boundary vertex sets in perfect graphs

3.1 General results

This section is devoted to studying the geodeticity of the boundary-type sets considered in the previous section, when restricting ourselves to several classes of perfect graphs [6] (see Figure 9). Firstly, we show some of the main known results concerning this issue.

Lemma 8 [3] *The contour of every graph $G = (V, E)$ is a hull set, i.e.,* $[\mathrm{Ct}(G)] = V$.

Lemma 9 [5] *The extreme set of every Ptolemaic graph $G = (V, E)$ is a hull set, i.e., $[Ext(G)] = V$.*

Lemma 10 [2] *The boundary of every graph $G = (V, E)$ is geodetic, i.e., $I[\partial(G)] = V$.*

Lemma 11 [3,8] *The contour of every distance-hereditary graph $G = (V, E)$ is geodetic, i.e., $I[Ct(G)] = V$.*

We will also need the following result.

Lemma 12 *Let G be a graph and let $u_0 \in V(G)$. Suppose that $(u_0, u_1, \dots, u_n)$ is a path in G such that* $\mathrm{ecc}(u_{i+1}) = \mathrm{ecc}(u_i) + 1$*, for each $i \in \{0, 1, \dots, n-1\}$. Then, for each eccentric vertex x of u_n, there exists a shortest path between x and u_n that contains u.*

PROOF. Let x be an eccentric vertex of u_n and suppose that $d(x, u_n) = \mathrm{ecc}(u_n) = k$. Then, by hypothesis, $\mathrm{ecc}(u_0) = k - n$. Let us prove that x is an eccentric vertex of u_0. Suppose, on the contrary, that $d(x, u_0) < k - n$ and consider a $x - u_0$ shortest path between them. Hence, $d(x, u_n) \leq d(x, u_0) + d(u_0, u_n) < k - n + n = k$, which is a contradiction with $d(x, u_n) = k$. Thus, $d(x, u_0) = k - n$ and the path $(x, \dots, u_0, u_1, \dots, u_n)$ is the desired shortest path between x and u_n.

3.2 Perfect graphs

We consider now the geodetic character of boundary vertex sets in perfect graphs. First, notice that the eccentric subgraph $\mathrm{Ecc}(G)$ of a graph G needs not to be a hull set, even if G is either bipartite or chordal graph; for example, the tree T displayed in Figure 5 satisfies: $\mathrm{Ecc}(T) = \{1, 5\}$ and $[\mathrm{Ecc}(T)] = V(T) \setminus 6$.

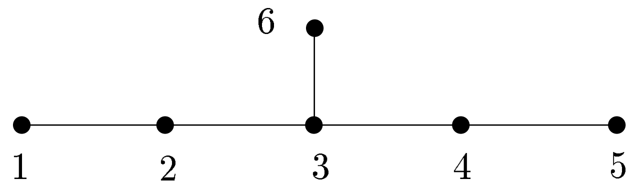

Figure 5. Tree T for which $[\mathrm{Ecc}(T)] \subsetneq V(T)$.

Secondly, observe that the contour $\mathrm{Ct}(G)$ of a perfect graph needs not to be geodetic; for example, the graph P illustrated in Figure 6 is a permutation graph (and hence perfect) which satisfies $\mathrm{Ct}(P) = \{1, 2, 3\}$ and $[\mathrm{Ct}(P)] = V(P) \setminus 4$.

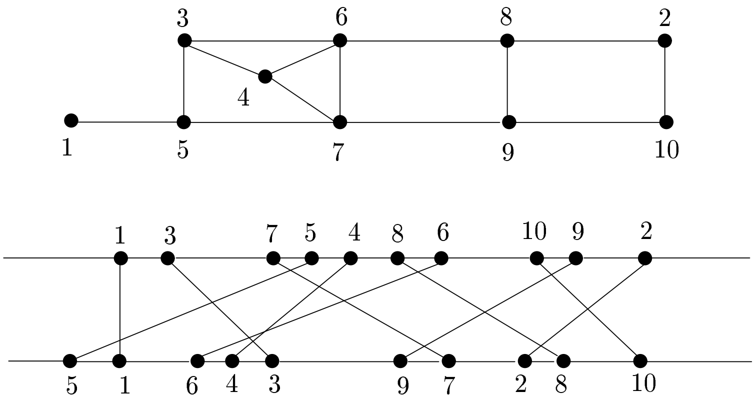

Figure 6. Permutation graph P and its matching diagram.

3.3 Chordal graphs

For the study of chordal graphs, we use their intersection representation. In general, for any family $\mathcal{F}$ of certain objects not necessarily different where intersection makes sense, the *intersection graph* $\Omega(\mathcal{F})$ is the graph with vertex set $\mathcal{F} = \{S_1, \ldots, S_n\}$ and such that two vertices $S_i, S_j \in \mathcal{F}$ are adjacent if and only if $i \neq j$ and $S_i \cap S_j \neq \emptyset$ [11]. For example, the intersection graph of the family of subsets $\mathcal{F} = \{S_1, S_2, S_3\}$, where $S_1 = \{x\}$, $S_2 = \{x\}$ and $S_3 = \{x, y\}$, is isomorphic to the complete graph K_3.

It is well known (see chapter 2 in [11]) that a connected graph G is chordal if and only if it is isomorphic to the intersection graph of a family of subtrees of a tree. Therefore, for any chordal graph G, there exists a tree T and a family of subtrees $\mathcal{F} = \{A(x) : x \in V(G)\}$ such that $xy \in E(G)$ if and only if $x \neq y$ and $V(A(x)) \cap V(A(y)) \neq \emptyset$, that is, G is isomorphic to the intersection graph $\Omega(\mathcal{F})$. The pair $(T, \mathcal{F})$ is called a *tree intersection representation* of G and T is the *host tree*. If $x \in V(G)$, we say that the subtree $A(x) \in \mathcal{F}$ *represents* the vertex x. For any vertex $u \in V(T)$, T_u denotes the trivial subtree of T formed only by vertex u.

Notice that neither the host tree T nor the family of subtrees $\mathcal{F}$ are unique. We may also assume that all vertices of T lie on at least a subtree of $\mathcal{F}$, since the removal of the vertices not satisfying this condition does not modify the structure of the intersection graph. The next lemma shows that every chordal graph G admits a tree intersection representation $(T, \mathcal{F})$ such that for every leaf u of the host tree T the trivial subtree T_u is in $\mathcal{F}$.

Lemma 13 *Every chordal graph G admits a tree intersection representation $(T, \mathcal{F})$ such that every leaf u of T belongs to a trivial subtree of $\mathcal{F}$.*

PROOF. For any intersection representation tree $(T, \mathcal{F})$ of a chordal graph G let us denote by $n(T, \mathcal{F})$ the number of leaves of T that do not belong to a trivial subtree of $\mathcal{F}$. Consider a tree intersection representation $(T, \mathcal{F})$ of G such that all vertices of T lie on at least a subtree of $\mathcal{F}$ with minimum $n(T, \mathcal{F}) \geq 0$. We have to prove that $n(T, \mathcal{F}) = 0$. Suppose, on the contrary, that $n(T, \mathcal{F}) > 0$ and let u_0 be a leaf of T such that the trivial subtree T_{u_0} is not in $\mathcal{F}$. If T is not a path, let $(u_0, u_1, \ldots, u_h, v)$, be the unique path in T such that $\deg_T(v) \geq 3$ and $\deg_T(u_i) = 2, \forall i \in \{1, 2, \ldots, h\}$. If T is a path, then $(u_0, u_1, \ldots, u_h, v)$, is the tree T.

We now modify the host tree T and the subtrees of $\mathcal{F}$ according to the following two cases obtaining a new tree intersection representation $(T', \mathcal{F}')$ of G with $n(T', \mathcal{F}') = n(T, \mathcal{F}) - 1 < n(T, \mathcal{F})$, which contradicts the fact that $n(T, \mathcal{F})$ is minimum.

(1) *There is no vertex $x \in V(G)$ represented by a subtree $A(x) \in \mathcal{F}$ contained in the path $P = (u_0, u_1, \ldots, u_h)$*: In this case, every subtree of $\mathcal{F}$ meeting P must contain v. Thus, by deleting the vertices $u_0, u_1, \ldots, u_h$ both from T and from the subtrees of $\mathcal{F}$ we obtain a new tree T' and a family of subtrees $\mathcal{F}'$ that keep all the intersection relations. That is, $(T', \mathcal{F}')$ is a tree intersection representation of G and, since $\deg_{T'}(v) \geq 2$, $n(T', \mathcal{F}') = n(T, F) - 1$ (Figure 7a).

(2) *There exists at least a vertex $x \in V(G)$ represented by a subtree $A(x) \in \mathcal{F}$ contained in the path $P = (u_0, u_1, \ldots, u_h)$*: Let $j \in \{1, \ldots, h\}$ be the minimum index such that there is a vertex $x \in V(G)$ represented by a subtree $A(x)$ contained in P. Thus, for some $i \in \{0, \ldots, h-1\}$ the path $(u_i, u_{i+1}, \ldots, u_j)$ is a subtree of $\mathcal{F}$ and all the subtrees containing any of the vertices $u_0, u_1, \ldots, u_{j-1}$ must also contain u_j. If we delete the vertices $u_0, u_1, \ldots, u_{j-1}$ both from T and from the subtrees of $\mathcal{F}$, we obtain a new tree T' and a family of subtrees $\mathcal{F}'$ that keep all the intersection relations. In addition, u_j is a leaf of T' and the subtree T_{u_j} is in $\mathcal{F}'$, implying that $n(T', \mathcal{F}') = n(T, \mathcal{F}) - 1$ (Figure 7b).

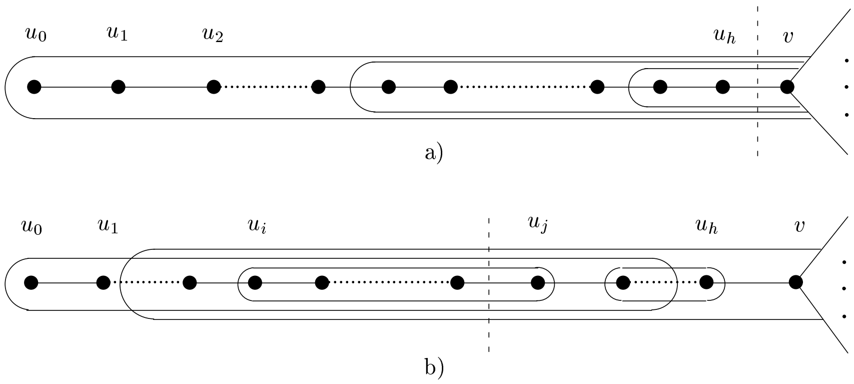

Figure 7. Modifying the host tree T and the family of subtrees $\mathcal{F}$.

In the rest of this paper, we assume that G is a chordal graph, and that $(T, \mathcal{F})$ is a tree intersection representation of G satisfying the statement of lemma 13.

Lemma 14 *For every leaf u of T, the vertex of G represented by T_u is an extreme vertex of G.*

PROOF. Let $x \in V(G)$ such that $A(x) = T_u$. All vertices adjacent to x are represented by subtrees meeting T_u, i.e., each of them contains the leaf u, which means that the neighborhood of x is a clique, or equivalently, that $V \setminus \{x\}$ is convex, as desired.

Lemma 15 *All vertices in G have at least an eccentric vertex which is represented by a trivial subtree T_w, where w is a leaf of T.*

PROOF. Given $x \in V(G)$, let e_x be an eccentric vertex of x, i.e., $d(x, e_x) = \mathrm{ecc}(x) = k$. Let $A(x)$ and $A(e_x)$ be the subtrees of T representing x and e_x, respectively. Assume that $u \in V(A(x))$, $v \in V(A(e_x))$ and consider the path on T between u and v. If v is not a leaf, we can extend the path from v to a leaf w (Figure 8). If v is a leaf, consider $w = v$. Thus, we obtain a path P of T, from u to a leaf w. Let $z \in V(G)$ be the vertex represented by the subtree $A(z) = T_w$.

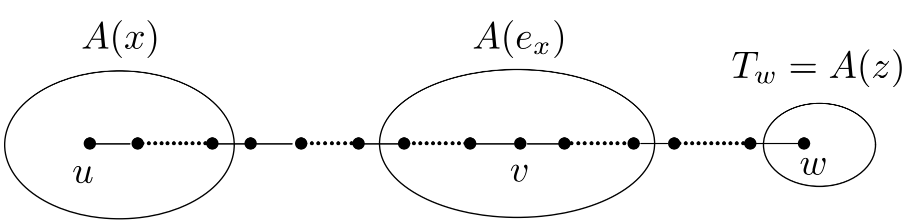

Figure 8. $A(z) = T_w$ is an eccentric vertex of $A(x)$.

Let us show that z is an eccentric vertex of x. Suppose, on the contrary, that $d(x, z) = h < k$. Rename $x = x_0$, $z = x_h$ and take a shortest path $(x_0, x_1, \ldots, x_h)$ in G. Notice that the subgraph L induced in T by vertices lying on the subtrees representing $x_0, x_1, \ldots, x_h$ is connected. Since P is the unique path of T joining $u \in V(A(x_0))$ and $z \in V(A(x_h))$, we have $V(P) \subseteq V(L)$. This means that $v \in V(L)$, so it lies on a subtree representing a vertex x_j, for some $j \in \{0, 1, \ldots, h\}$. Thus, we have $v \in V(A(x_j)) \cap V(A(e_x))$, which implies that x_j and e_x are adjacent.

If $j < h$ there exists a path $(x_0, x_1, \ldots, x_j, e_x)$ of length $j+1 \leq h < k = \mathrm{ecc}(x)$, which contradicts that e_x is an eccentric vertex of x. Hence, we have that $j = h$, that is, v lies on the subtree $T_w = \{w\}$ representing $x_h = z$, so that $v = w$. Furthermore, since x_{h-1} and $x_h = z$ are adjacent vertices in G, v lies also on the subtree representing x_{h-1}, which meets T_w, and we can build a path $(x_0, x_1, \ldots, x_{h-1}, e_x)$ of length $h < k$, which is again a contradiction. Therefore, $d(x, z) = k = \mathrm{ecc}(x)$, and z is an eccentric vertex of x, which is represented by a leaf of T.

Finally, we use the lemmas above to prove our main result.

Theorem 16 *In every chordal graph, the contour is geodetic.*

PROOF. Let G be a chordal graph and let $x_0 \in V(G)$. If $x_0 \notin \mathrm{Ct}(G)$, then it has a neighbor x_1 with greater eccentricity. We repeat this argument to obtain a path $(x_0, x_1, \ldots, x_k)$ where $\mathrm{ecc}(x_{i+1}) = \mathrm{ecc}(x_i) + 1$, $\forall i \in \{0, 1, \ldots, k-1\}$, and x_k is a contour vertex. According to Lemma 15, there exists an eccentric vertex of x_k, say z, which is represented by a leaf, and by Lemma 14, such a vertex is a contour vertex. Finally, using Lemma 12, we conclude that there exists a shortest path between x_k and z containing x_0, as desired.

Certainly, every convex set of a chordal graph induces a chordal subgraph. As a direct consequence, we obtain the following theorem.

Theorem 17 *If G is a chordal graph, then every convex set of G is the geodetic closure of its contour.*

4 Conclusions and open problems

We have presented a realization theorem involving the cardinalities of the periphery, contour, eccentric subgraph and boundary of a chordal graph, which solves completely the approached problem (see Theorem 6). We have also showed a similar result by considering, instead of the contour, the extreme set (see Theorem 7). Contrarily to Theorem 6, Theorem 7 does not completely resolve the posed problem since the constraint $2 \leq a \leq \min\{b, c\} \leq \max\{b, c\} \leq d$ is not sharp. Hence, it remains to be proved or disproved a similar result to the previous one after replacing the mentioned constraint by these ones: $2 \leq a \leq c \leq d$, $2 \leq b \leq d$.

It has recently been proved both that the boundary of every graph is geodetic, and that the contour of every distance-hereditary is geodetic. Starting from these facts, we have studied the geodeticity of the more significant subsets of the boundary (extreme set, periphery, contour and eccentric subgraph) for the class of perfect graphs introduced by Claude Berge [1]. First, we have showed that neither the contour nor the eccentric subgraph of a perfect graph need no to be geodetic. And finally, we have proved that the contour of every chordal graph is geodetic. Figure 9 shows the Hasse diagram of the perfect graphs and some of its sub-families, showing the current state of knowledge on the topic: *Geodeticity of the contour*, for each subfamily. Note that the only perfect families in this diagram for which the question is still open are the co-chordal, parity and bipartite graphs.

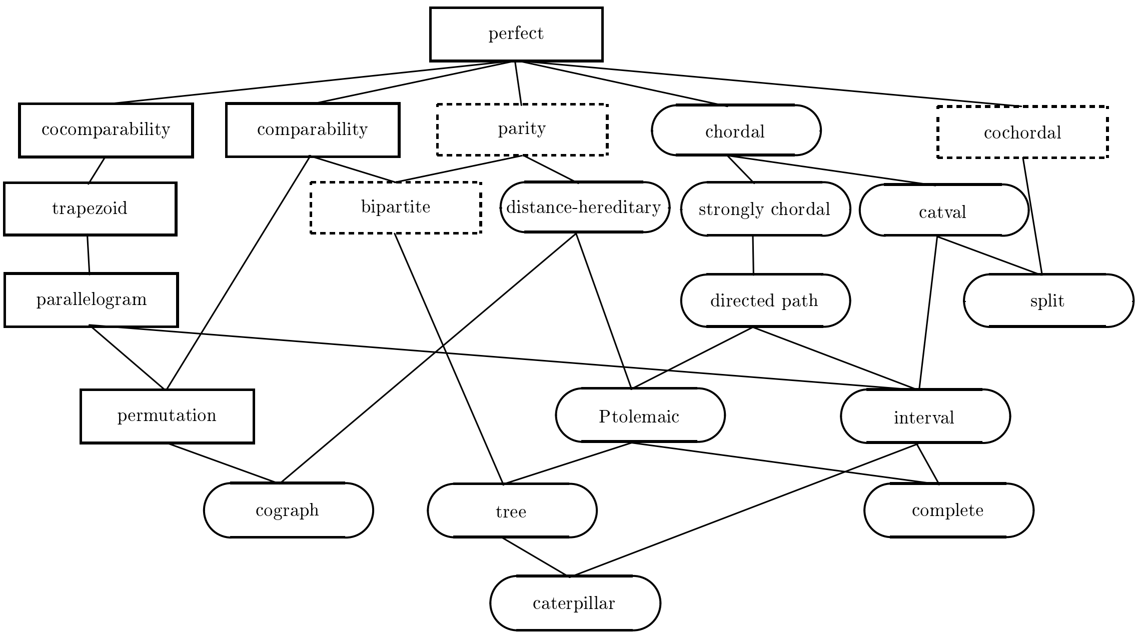

Figure 9. Is the contour geodetic?: Yes for solid ovals, not necessarily for solid squares, and open problem for dashed squares.

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