International Journal of Computational Geometry & Applications

ON COMPUTING ENCLOSING ISOSCELES TRIANGLES AND RELATED PROBLEMS*

PROSENJIT BOSE
School of Computer Science, Carleton University, 1125 Colonel By Drive
Ottawa, Ontario K1S5B6, Canada
jit@scs.carleton.ca

MERCÈ MORA
Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Jordi Girona 1
Barcelona, 08034, Spain
merce.mora@upc.edu

CARLOS SEARA
Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Jordi Girona 1
Barcelona, 08034, Spain
carlos.seara@upc.edu

SAURABH SETHIA
saurabhsethia@gmail.com

Received (received date)
Revised (revised date)
Communicated by (Name)

Given a set of n points in the plane, we show how to compute various enclosing isosceles triangles where different parameters such as area or perimeter are optimized. We then study a 3-dimensional version of the problem where we enclose a point set with a cone of fixed apex angle α.

Keywords: Enclosing isosceles triangle; perimeter; area.

1. Introduction

Given a set $P = \{p_1, \ldots, p_n\}$ of n points in two or three dimensions, the problem of computing a geometric structure enclosing the point set while optimizing some criteria of the enclosing structure such as area, perimeter, surface area or volume has been widely studied in the literature [2,3,4,5,6,7,8,10,12,15,17,19,20]. In this paper, we are particularly interested in the two dimensional setting where the enclosing structure is a triangle. The two natural parameters to optimize in this setting are

*Partially supported by projects MEC MTM2006-01267 and DURSI 2005SGR00692.

the area or the perimeter of the enclosing triangle. Both problems are well-studied in the literature. For the former, Klee and Laskowski [12] presented an $O(n \log^2 n)$ time algorithm to compute the enclosing triangle of minimum area and O'Rourke et al. [17] improved this to linear, which is optimal. For the latter, De Pano [10] proposed an $O(n^3)$ algorithm for computing the enclosing triangle of minimum perimeter. This was subsequently improved by Chang and Yap [8] as well as Aggarwal and Park [3] culminating in the optimal linear time algorithm of Bhattacharya and Mukhopadhyay [4]. In this paper, we study several variants of the problem where the enclosing structure is not an arbitrary triangle, but an isosceles triangle T. As we shall see, this constraint changes the problem. For convenience, we will denote the *apex* of an isosceles triangle T by a; the angle at the apex, *the apex angle*, by α; the edge opposite the apex referred to as the *base edge* by b; and the distance from the apex to the base edge which is the *height* by h (Figure 1). The *orientation* of T is given by a unit vector of the oriented bisector line of α, from base edge to apex, i.e., by a unit vector perpendicular to the base edge, from b to a. The bisector line of the apex angle α is the axis of T.

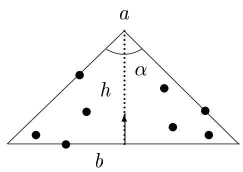

Figure 1. Notation for isosceles triangle.

The natural criteria to optimize when computing an enclosing isosceles triangle are the apex angle α, the height h, the perimeter, or the area. Given an isosceles triangle of fixed apex angle α, $0 < \alpha < \pi$, one can observe the following.

Observation 1. *The enclosing isosceles triangle for P with fixed apex angle α of minimum height h also has minimum area and minimum perimeter.*

$$\text{Perimeter} = 2h\left(\tan\frac{\alpha}{2} + \sqrt{1+\tan^2\frac{\alpha}{2}}\right) \qquad \text{Area} = h^2\tan\frac{\alpha}{2}$$

This is a key difference between the optimization problem with isosceles triangles versus arbitrary triangles. This leads to the first optimization problem we study:

Problem 1. *Given P and a positive constant $0 < \alpha < \pi$, compute the enclosing isosceles triangle for P with apex angle α and minimum height h.*

Let $w(P)$ be the width of the point set P [11], i.e., the minimum distance between parallel lines of support passing through an antipodal vertex-edge pair of the convex hull of P, $CH(P)$. If the height h, $h \geq w(P)$, of the enclosing isosceles triangle for P is fixed, one can observe:

Observation 2. *The enclosing isosceles triangle for P with fixed height h of minimum apex angle α, $0 < \alpha < \pi$, also has minimum area and minimum perimeter.*

This leads to the second optimization problem we study:

Problem 2. *Given P and a fixed height h, $h \geq w(P)$, compute the enclosing isosceles triangle for P with height h and minimum apex angle α.*

We end with an exploration of a 3-dimensional variant of this problem. In this setting, the enclosing structure is a cone with angle α at the apex and a circular base, which we call an α-cone. We solve 3-dimensional versions of Problems 1 and 2 with the additional constraint that the orientation of the α-cone is fixed. The general 3-dimensional problem is more difficult since there are more degrees of freedom for the problem. We elucidate some of these difficulties and conclude with a few open problems.

2. Enclosing isosceles triangle

Prior to solving Problems 1 and 2, we first investigate a simpler subproblem which will prove to be useful in the solutions of the other two problems. For simplicity, we assume that all point in P are in general position.

2.1. *Simpler subproblem*

Given P, we wish to compute an enclosing isosceles triangle for P with apex angle α, horizontal base, and minimum height. Since the apex angle is fixed and the base is horizontal, the two edges of the enclosing isosceles triangle adjacent to the apex have fixed directions. Both these edges must be tangent to $CH(P)$. Both these tangents can be computed in $O(n)$ time by computing the supporting lines forming the corresponding angles with the horizontal line. If the input set is a convex polygon, then both supporting lines can be computed in $O(\log n)$ time [16].

Proposition 1. *The enclosing isosceles triangle for P with fixed apex angle, horizontal base, and minimum height can be computed in $O(n)$ time.*

This simple result immediately gives rise to an approximation scheme.

2.2. *Problem 1: PTAS for minimum height*

In the following we describe a Polynomial Time Approximation Scheme for Problem 1. Given P and a fixed angle $0 < \alpha < \pi$, our algorithm finds an enclosing isosceles triangle with apex angle α whose height is within a multiplicative factor $(1 + \epsilon)$ of the optimal height. The running time is $O(n)$ with a constant that depends on α. As α tends to 0 or π, the running time tends to infinity. We outline the algorithm below.

PTAS-algorithm

(1) Choose a set D of orientations on the unit circle so that any orientation d on the unit circle has some orientation $d^* \in D$ where the angle between d and d^* is less than δ, with

$$\sin\delta = \frac{\epsilon \sin\alpha}{2-\cos\alpha}.$$

This can be accomplished using $O(\delta^{-1})$ uniformly spread orientations.

(2) For each orientation in D, compute the enclosing isosceles triangle for P using Proposition 1. Return the triangle of minimum height among all computed enclosing isosceles triangles.

To prove the correctness of this algorithm we will need the following lemma.

Lemma 1. *Let T be an isosceles triangle with apex angle α and height h. Let T' be the isosceles triangle of minimum height h' containing T, with apex angle α, and base edge tilted by an angle of δ. Then, we have that*

$$h' = h\left(\frac{(2-\cos\alpha)\sin\delta}{\sin\alpha} + \cos\delta\right).$$

Proof. Figure 2 shows an isosceles triangle T of height h inside another isosceles triangle T' tilted by an angle of δ. Using elementary trigonometry we can calculate the height h' of T' as follows. By $\overline{AD}$ we denote the length of segment AD and by β we denote $\frac{\alpha}{2}$.

$$\overline{XW} = \overline{XZ}\cos\beta = \left(\overline{XA} + \overline{AP} + \overline{PZ}\right)\cos\beta =$$

$$= \left(\frac{\overline{AQ}}{\sin\alpha} + \overline{AC}\cos\delta + \overline{PC}\tan\beta\right)\cos\beta =$$

$$= \left(\frac{\overline{AB}\sin\delta}{\sin\alpha} + \frac{\overline{AD}\cos\delta}{\cos\beta} + \overline{AC}\sin\delta\tan\beta\right)\cos\beta =$$

$$= \left(\frac{\overline{AD}\sin\delta}{\cos\beta\sin\alpha} + \frac{\overline{AD}\cos\delta}{\cos\beta} + \frac{\overline{AD}\sin\delta\tan\beta}{\cos\beta}\right)\cos\beta =$$

$$= \frac{\overline{AD}\sin\delta}{\sin\alpha} + \overline{AD}\cos\delta + \overline{AD}\sin\delta\tan\beta =$$

$$= \overline{AD}\left(\frac{\sin\delta}{\sin\alpha} + \cos\delta + \sin\delta\tan\beta\right) = \overline{AD}\left(\sin\delta\left(\frac{1}{\sin\alpha} + \tan\beta\right) + \cos\delta\right) =$$

$$= \overline{AD}\left(\sin\delta\left(\frac{1}{2\sin\beta\cos\beta} + \frac{\sin\beta}{\cos\beta}\right) + \cos\delta\right) =$$

$$= \overline{AD}\left(\sin\delta\left(\frac{1+2\sin^2\beta}{2\sin\beta\cos\beta}\right)+\cos\delta\right)=$$

$$= h\left(\sin\delta\left(\frac{1+2\sin^2\beta}{\sin\alpha}\right)+\cos\delta\right)= h\left(\frac{\left(1+2\sin^2\frac{\alpha}{2}\right)\sin\delta}{\sin\alpha}+\cos\delta\right)=$$

$$= h\left(\frac{(2-\cos\alpha)\sin\delta}{\sin\alpha}+\cos\delta\right).$$

□

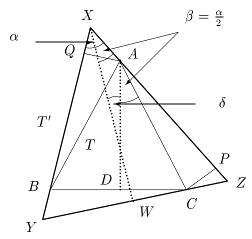

Figure 2. Isosceles triangle T inside the isosceles triangle T' tilted by an angle of δ.

Theorem 1. *The algorithm returns an enclosing isosceles triangle for P with apex angle α whose height is at most $(1+\epsilon)$ times the optimal height. The running time of the algorithm is $O(n)$ for any fixed α, such that $0 < \alpha < \pi$.*

Proof. Consider the axis orientation d of the optimal isosceles triangle T. At least one orientation d' of set D will be within δ of d, where δ is as defined in Lemma 1. The optimal isosceles triangle T' over T with axis along d' will also be an enclosing isosceles triangle for P. Hence the height of T' is an upper bound on the output of the algorithm. From Lemma 1 and our choice of δ this height is less than or equal to $h(\cos\delta + \epsilon)$ which is less than or equal to $h(1+\epsilon)$.

The algorithm solves the *simpler subproblem* $O(\delta^{-1})$ times using Proposition 1. Since δ^{-1} is constant for $0 < \alpha < \pi$, the running time of the algorithm is $O(n)$. □

We continue by developing an exact algorithm whose running time increases by a log factor. However, we will show that this is optimal by providing a lower bound.

2.3. *Problem 1: minimum height*

In this section we solve the first problem exactly. Recall that the input is a set P of n points in the plane and we want to compute the enclosing isosceles triangle with fixed apex angle α that has minimum height. Observe that the optimal solution must have at least one point of P on each edge of the triangle. This reduces the search space for the optimal solution. In two dimensions, we define an α-wedge, for $0 < \alpha < \pi$, to be the set of points contained between two rays emanating from the same point called the apex, with the angle between the two rays being α. We say that an α-wedge is *minimal* provided that $P \subset \alpha$-wedge and each of the rays contains at least one point of P. The orientation of the α-wedge is given by the oriented bisector line of α. As a first step, we compute the locus of points L with the property that for each point x of L, a minimal α-wedge exists with apex at x.

Theorem 2. *The locus L of apices of minimal α-wedges for P can be computed in $O(n \log n)$ time. If P is simple polygon the locus can be computed in $O(n)$ time. Furthermore, there is a bijection between point of the locus and orientation in the unit circle.*

Proof. To compute L, we first compute $CH(P)$. Note that for each orientation of an α-wedge, there is exactly one position for which it is minimal. Given a fixed orientation, if the minimal α-wedge for P only contains $p_i \in CH(P)$ on one ray and $p_j \in CH(P)$ on the other ray, as we rotate the orientation slightly in a clockwise or counter-clockwise fashion, these two points of contact do not change for minimal α-wedges in those orientations. It is not until a second point of $CH(P)$ touches one of the two rays does the contact point change. Therefore, for a fixed pair of contact points p_i and p_j, the locus of apices of minimal α-wedges forms an arc of a circle since α is fixed (Figure 3a). By modifying the rotating calipers algorithm [16,21], we can rotate the α-wedge around the convex hull and compute the sequence of at most linear number of circular arcs, which we will call an α-*cloud* [20], representing the set L (Figure 3b).

It is easy to see that there exists a bijection between points in L and orientations in the unit circle $\mathbb{S}^2$. Thus, each circular arc corresponds to an interval of orientations on $\mathbb{S}^2$; in fact, we will use the same notation $[A, B]$ for both. □

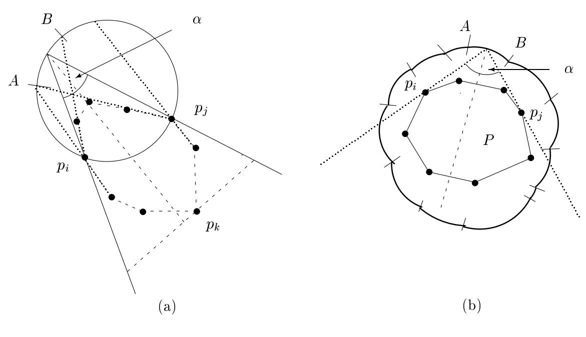

Figure 3. (a) Arc of circle for the points p_i and p_j, (b) α-cloud for P.

We next show how to compute an enclosing isosceles triangle for P of minimum height where the apex is constrained to lie on a circular arc of the α-cloud. Each of the sides of the enclosing triangle adjacent to the apex is in contact with at least one point of $CH(P)$. Let p_i and p_j be these contact points. As the apex slides over the circular arc, the base edge of the isosceles triangle is also in contact with a point, say p_k. This third point may change multiple times.

Lemma 2. *Let $[A, B]$ be a circular arc of the α-cloud where the contact points of each of the edges of the enclosing triangle are $p_i, p_j, p_k \in P$, respectively. The height of the enclosing isosceles triangle is a continuous and unimodal function over $[A, B]$. This height function has a constant number of maxima and minima.*

Proof. Let $[A, B]$ be a circular arc of the α-cloud where the contact points of each of the edges of the enclosing triangle are $p_i, p_j, p_k \in CH(P)$, respectively (Figure 4). As we slide the apex a of the α-wedge from A to B the bisector of α always pass through the point M which is the midpoint of the circular arc p_ip_j. Denote by N the intersection point of the bisector of α and the base edge of the isosceles triangle. Thus, the height is $h = \overline{aM} + \overline{MN}$. We will see that h is a function of the rotation-angle of the bisector as a moves from A to B.

First, without loss of generality, assume that $[A, B]$ contains the point D which defines the diameter $\overline{DM}$ of the circle with center C (Figure 4a). Notice that $\overline{aM} = \overline{DM} \cos\gamma$, where γ is the angle formed by the bisector and $\overline{DM}$, with γ being negative as the apex moves in $[B, D]$, being 0 in D, and being positive in $[D, A]$. The maximum value of $\overline{aM}$ is $\overline{DM}$. Thus, the function distance $\overline{aM}$ for $a \in [A, B]$ is a unimodal function of the angle γ.

Second, notice that M, N, and p_k form a right angled triangle because the perpendicularity of $\overline{Np_k}$ and $\overline{MN}$ (Figure 4b). As we move the apex a from A to B, the point N moves along a circular arc of the circle with diameter $\overline{Mp_k}$. Assume that $[A, B]$ also contains the point E defined by the intersection of $[A, B]$ with the line containing $\overline{Mp_k}$. Then $\overline{MN} = \overline{Mp_k} \cos\delta$, where δ is the angle formed by the bisector with $\overline{Mp_k}$, with δ being negative as the apex moves in $[B, E]$, being 0 in E, and being positive in $[E, A]$. The maximum value of $\overline{MN}$ is $\overline{Mp_k}$. Thus the function distance $\overline{MN}$ for $a \in [A, B]$ is a unimodal function of the angle δ.

Third, assume a general case where the relative position of the contact points is as illustrated in Figure 5, other cases can be handle analogously. We split $[A, B]$ in three consecutive sub-arcs: $[A, E]$, $[E, D]$, and $[D, B]$. In $[A, E]$, the height h is an increasing function because both distances $\overline{aM}$ and $\overline{MN}$ are increasing. In $[D, B]$, the height h is a decreasing function because both distances $\overline{aM}$ and $\overline{MN}$ are decreasing. In $[E, D]$, $\overline{aM}$ is increasing and $\overline{MN}$ is decreasing, nevertheless the height h can be formulated as follows:

$$h = \overline{aN} = \overline{aM} + \overline{MN} = \overline{DM}\cos(-\theta) + \overline{Mp_k}\cos(-\theta + \theta_0), \quad 0 < \theta_0 < \pi/2$$

$$h = (\overline{DM} + \overline{Mp_k} \cdot \cos\theta_0) \cdot \cos\theta + (\overline{Mp_k} \cdot \sin\theta_0) \cdot \sin\theta = c_1 \cos\theta + c_2 \sin\theta$$

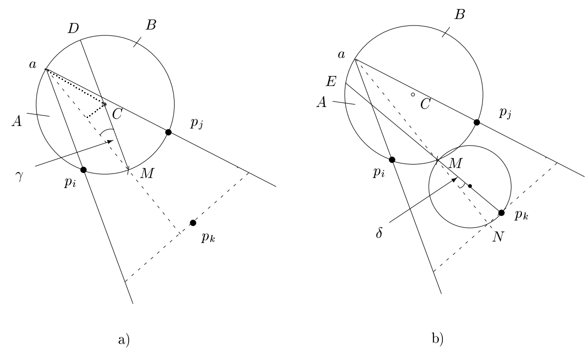

Figure 4. Decomposing the height of the α-wedge.

where c_1 and c_2 are positive constant. Taking derivatives of the function, we check that it has at most a maximum for the value θ_1 such that $\tan\theta_1 = c_2/c_1$, which can be computed in constant time. The local minima are in the extremes E, D.

We conclude that the variation of the height as the apex moves in $[A, B]$ is a continuous function. For each one of the arcs (or sub-arcs) $[A, D]$, $[D, E]$, and $[E, B]$ the height function has a maximum and at most two minima on the extremes of the arc (or sub-arc). Thus, in constant time, we can compute the minimum and maximum values of the height of the α-wedges with contact points p_i, p_j, p_k. $\square$

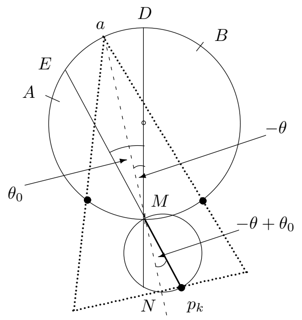

Figure 5. Computing the height in a general case.

Remark 1. We can consider the case where the base edge has consecutive contact points, say $p_k, p_{k+1}, p_{k+2}, \ldots$, while apex a moves from A to B. In this case we split the arc $[A, B]$ into consecutive sub-arcs, one per point, and proceed analogously as above. Notice that we only have to change the diameter $\overline{Mp_k}$ by $\overline{Mp_{k+1}}$ and so on.

Theorem 3. *Given P and α, $0 < \alpha < \pi$, the enclosing isosceles triangle for P with* *apex angle α and minimum (or maximum) height, minimum area, and minimum perimeter can be computed in $O(n \log n)$ time. If P is a simple polygon, the running time is reduced to $O(n)$.*

Proof. An enclosing isosceles triangle for P with fixed apex angle α can be seen as a three-sides caliper. A counting argument shows that the total number of triples of contact points examined over the whole α-cloud is linear. Since for each triple, the solution can be found in constant time, the overall complexity of the algorithm is $O(n \log n)$. Recall Observation 1 for minimum area or perimeter. □

2.4. *Problem 2: minimum angle*

In this setting, we wish to compute an enclosing isosceles triangle for P of fixed height h, and minimize the apex angle. If $w(P) > h$, then the problem has no solution. If $w(P) = h$ then the solution corresponds to two supporting parallel lines of $CH(P)$.

In what follows we assume that $w(P) \leq h$. We define the *width of P with orientation $\vec{u} \in \mathbb{S}^2$, $w_{\vec{u}}(P)$*, as the distance of the parallel lines of support for P which are perpendicular to $\vec{u}$. This distance over antipodal vertex-vertex pair of $CH(P)$ is an increasing/decreasing function with respect to the rotation angle. Let $I_h = \{\vec{u} \in \mathbb{S}^2 \mid w_{\vec{u}}(P) \leq h\}$. Clearly, I_h consists of a disjoint union of orientation intervals. Houle and Toussaint [11] use the rotating calipers technique for computing $w(P)$ in $O(n \log n)$ time for P or in $O(n)$ time for a simple polygon. With this technique, we can compute the variation of $w_{\vec{u}}(P)$ for $\vec{u} \in \mathbb{S}^2$. Given h, we can compute I_h as the disjoint union of orientation intervals on $\mathbb{S}^2$ in $O(n \log n)$ time for P or in $O(n)$ time for a simple polygon.

Lemma 3. *I_h can be computed in $O(n \log n)$ time for P or in $O(n)$ time for a simple polygon.*

First, we solve the simpler problem of computing the minimum apex angle if the orientation and height of the enclosing isosceles triangle for P are fixed.

Lemma 4. *The unique enclosing isosceles triangle for P with height h, horizontal base, and minimum apex angle can be computed in $O(n \log n)$ time. If P is a simple polygon, the running time is reduced to $O(n)$.*

Proof. Suppose that $(0, 1) \in I_h$. The unique enclosing isosceles triangle T for P with height h and horizontal base can be computed by the following algorithm.

(1) Compute $CH(P)$.
(2) Let p_b (p_t) be the bottom (top) point in the increasing y-coordinate order of $CH(P)$. Let p_l (p_r) be the left (right) most point of $CH(P)$ in the x-coordinate order. These four points can be computed in $O(\log n)$ time from $CH(P)$. The apex a of T lies in between the points A and B of the line ℓ as in Figure 6,

otherwise the bisector line on a is not a vertical line. We split the upper chain of $CH(P)$ into two chains: let L_c (R_c) be the upper chain in between p_l (p_t) and p_t (p_r). Notice that a contact point of T belongs to L_c and the other one belongs to R_c. The sizes of L_c and R_c can be linear.

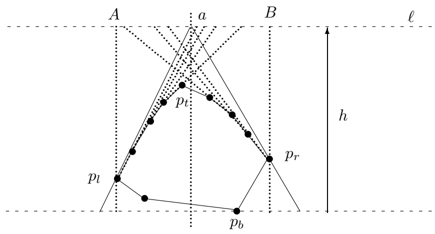

Figure 6. Isosceles triangle with horizontal base with fixed height h.

(3) Next, we determine an interval $[C, D]$ on ℓ containing the apex a of T such that C and D define the left and right contact points of T. Initially $[C, D] := [A, B]$. Let e_i be an edge in R_c, let r_i be the line containing e_i, and let c_i be the intersection point between ℓ and r_i. Choose $e_i \in R_c$ such that c_i is within the current $[C, D]$. In $O(\log n)$ time compute the supporting line l_i of $CH(P)$ passing through c_i and touching L_c; let s_i be the slope of the bisector of the apex angle defined by l_i and r_i. If s_i is negative (positive) do $C := c_i$ ($D := c_i$). Use binary search to find a new edge $e \in R_c$ such that the intersection point between the line containing e and ℓ is within $[C, D]$. Do $e_i := e$ and proceed as above until the extremes C, D do not change anymore.

(4) From the last $[C, D]$ and the contact points, says p_i and p_j, we proceed as follows. Assume that the horizontal line passing through p_b is the x-axis. Let $a = (x, h)$, $C \leq x \leq D$, $p_i = (x_i, y_i)$, and $p_j = (x_j, y_j)$. In constant time compute a and α with the equations (1).

$$\tan\frac{\alpha}{2} = \frac{x - x_i}{h - y_i} = \frac{x_j - x}{h - y_j}, \qquad \tan\frac{\alpha}{2} = \frac{x_j - x_i}{2h - (y_i + y_j)}. \tag{1}$$

Step 1 can be done in $O(n \log n)$ time, if P is a simple polygon we can do this step in $O(n)$ time. Step 2 can be done in $O(\log n)$ time. Step 3 can be done $O(\log^2 n)$ time because we do a double binary search. Step 4 only needs constant time. The uniqueness of the enclosing isosceles triangle comes from the unique solution of the equations (1). □

Remark 2. The step 3 of the above Lemma 4 can be done as follows. Instead of doing a double binary search to determine the interval $[C, D]$ we take adjacent edges $e_i \in R_c$ and for each one we compute the intersection point c_i of r_i and ℓ and the corresponding supporting line l_i from c_i. The complexity of this step 3 is $O(n \log n)$. But the overall running time of the algorithm in Lemma 4 for set of points P is still $O(n \log n)$. We use this remark later.

Next we consider the relationship between the apex angles for different heights.

Lemma 5. *Let T_1 and T_2 be the enclosing isosceles triangles for P for a fixed orientation, and let h_1, α_1 and h_2, α_2 be the respective heights and apex angles. Then $\alpha_2 < \alpha_1$ if and only if $h_2 > h_1$.*

Proof. Assume that the fixed orientation is the vertical. Let $p_i = (x_i, y_i)$ and $p_j = (x_j, y_j)$ be the contact points for both T_1 and T_2. If the segment $\overline{p_i p_j}$ is horizontal, the axes of T_1 and T_2 are on the same vertical line, otherwise the axis of T_2 is on the left (right) of the axis of T_1 if $y_i < y_j$ ($y_i > y_j$) (Figure 7). The heights h_1 and h_2 can be computed from equations (1). It follows that:

$$\alpha_2 < \alpha_1 \Longleftrightarrow \tan\frac{\alpha_2}{2} < \tan\frac{\alpha_1}{2} \Longleftrightarrow \frac{x_j - x_i}{2h_2 - (y_i + y_j)} < \frac{x_j - x_i}{2h_1 - (y_i + y_j)} \Longleftrightarrow h_2 > h_1.$$

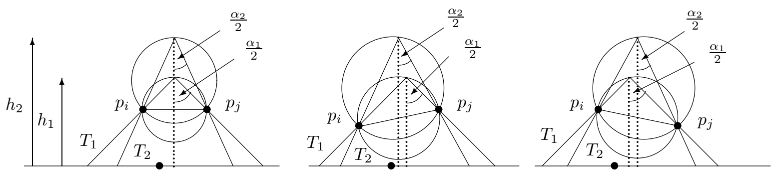

Figure 7. Enclosing isosceles triangles with horizontal base and heights h_1 and h_2.

Now, suppose that the apex angle decreases from α_1 to α_2 and a contact point of T_1 changes from p_j to $p_{j+1} = (x_{j+1}, y_{j+1})$, where $x_{j+1} > x_j$ and $y_{j+1} < y_j$ (Figure 8). The following formula holds.

$$\tan\frac{\alpha_2}{2} = \frac{x_j - x_i}{2h_2 - (y_i + y_j)} = \frac{x_{j+1} - x_i}{2h_2 - (y_i + y_{j+1})}.$$

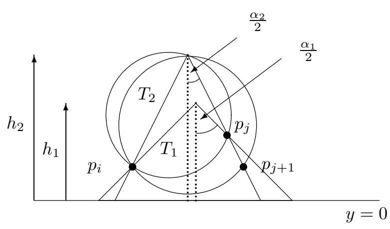

Figure 8. Enclosing isosceles triangles with contact point p_j and p_{j+1}.

It follows that $\alpha_2 < \alpha_1$ if and only is $h_2 > h_1$. Thus, we can proceed in discrete steps (decreasing the apex angle) as the sides of the enclosing isosceles triangles for P bumps new contact points and obtaining a sequence of equivalences which prove the desired result. □

In addition to the property outlined in Lemma 2, we exploit the following two additional properties of α-clouds which are straightforward consequences from Lemmas 2 and 5.

Properties 1. *Given two different apex angles α_1 and α_2:*

(1) If $\alpha_1 < \alpha_2$, then the α_2-cloud for P is contained in the α_1-cloud for P (Figure 9).
(2) The minimum height of the enclosing isosceles triangles for P with apex angle α_2 and apex on the α_2-cloud is at most the minimum height of all the enclosing isosceles triangles for P with apex angle α_1 and apex on the α_1-cloud.

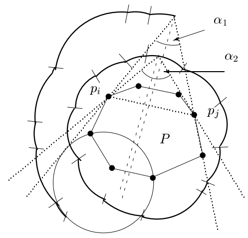

Figure 9. Locus of apices of α_1-wedges and α_2-wedges.

We use these properties to design an algorithm for computing the minimum apex angle α such that the minimum height of all the enclosing isosceles triangles for P with apices on the α-cloud is exactly h. To initialize the algorithm we compute an initial solution, say α_0, with the given height h. We concentrate on the circular arc (or sub-arc) of the α_0-cloud for P where the height is minimum. Then, we focus on orientations on the (clockwise sorted) circular arcs or sub-arcs of α_1-clouds for P where the height is smaller than h, by decreasing the angle α_1 till we get the height h and maintaining always a witness orientation with height h. By exploiting the properties of clouds outlined above and considering that an enclosing isosceles triangle is a type of caliper, we are able to extract the enclosing isosceles triangle with height h and minimum apex angle for all the orientations in $\mathbb{S}^2$ with the following algorithm.

Minimum-apex-angle-fixed-height-algorithm

(1) *Preprocess*. In $O(n \log n)$ time compute $CH(P)$ and sort the points of $CH(P)$

in both clockwise and counter-clockwise order. By Lemma 3, in $O(n)$ time we compute I_h. Assume that the vertical orientation $\vec{v} = (0,1) \in I_h$. The next steps of the algorithm are restricted to orientations in I_h.

(2) *Initial step.* By Lemma 4, in $O(n)$ time we compute the enclosing isosceles triangle for $CH(P)$ with horizontal base and height h. Let α_0 be its apex angle. By Theorem 3, in $O(n)$ time compute the α_0-cloud and the enclosing isosceles triangle T_0 for $CH(P)$ with apex in the α_0-cloud and minimum height, say h_0, $w(P) \leq h_0 \leq h$. If $h_0 = h$, we are done.

Suppose that $h_0 < h$. By Theorem 3 the height h_0 corresponds to an extreme, say B_0, of an *arc* (or *sub-arc*) of interval orientation $[A_0, B_0]$ of the α_0-cloud with a triple of contact points. Let $\vec{u} := B_0$, $\vec{u} \in I_h$. The orientation (or apex) $\vec{u} := B_0$ is the starting point of the next iteration step.

(3) *Clockwise-rotational process.* Compute the enclosing isosceles triangle T_1 with orientation $\vec{u}$ and height h as follows:

(a) By Lemma 5, we increase the height by decreasing the apex angle as we take (left and right) edges incident to the current contact points. We use Remark 2 and Lemma 4 to compute the apex angle α_1 of the enclosing isosceles triangle T_1 with height h and orientation $\vec{u}$ by taking adjacent edges on the right side of the isosceles triangle, and charging the $O(\log n)$ time complexity to each new (right) edge we use. To compute the apex angle α_1 (smaller than the current apex angle) and the apex A_1 of T_1 for orientation $\vec{u}$ we use the equations (1).

(b) In constant time compute the next (clockwise) arc or sub-arc $[A_1, B_1]$ of the α_1-cloud as orientation interval in $\mathbb{S}^2$ and the height h_1 of the enclosing isosceles triangle with apex (or orientation) B_1.

i. If $h_1 \geq h$, then proceed as in step 3(b) while we do a complete round in $\mathbb{S}^2$, otherwise go to step 4.

ii. If $h_1 < h$, then do $\vec{u} := B_1$ and go to step 3 to compute the new apex angle α_1.

(4) *Final step.* Let α_1 be the last computed value for the apex angle after a complete round of orientations in $\mathbb{S}^2$. In linear time we compute the α_1-cloud for $CH(P)$. Then the minimum height for the enclosing isosceles triangles with apices on this α_1-cloud is h and it can only be obtained in the extremes of the arcs or sub-arcs of the cloud, and therefore at most a linear number of times.

Theorem 4. *Given P and a fixed value h, the enclosing isosceles triangle with height h and minimum apex angle (if it exists), minimum perimeter, and minimum area, can be computed in $O(n \log n)$ time.*

Proof. The algorithm above solves the problem in $O(n \log n)$ time. Step 1 can be done in $O(n \log n)$. By Lemma 4 and Theorem 3, step 2 can be done in $O(n)$. Step 3 can be done in $O(n \log n)$ time since we charge the $O(\log n)$ complexity to the (right) visited edges in the clockwise rotational process, as a caliper which rotates clockwise each time with smaller or equal apex angle. Finally, by Theorem 3, step 4 can be done in $O(n)$ time. Recall the Observation 2 for minimum area or perimeter. $\square$

2.5. *Lower bounds*

Theorem 5. *Given P and α, $0 < \alpha < \pi$, computing the enclosing isosceles triangle for P with apex angle α and minimum height, minimum perimeter, and minimum area, requires $\Omega(n \log n)$ operations in the algebraic computation tree model.*

Proof. We use a reduction to Max-Gap problem for points in the first quadrant on the unit circle [13,18] to show an $\Omega(n \log n)$ time lower bound for the problem of computing the minimum height of an enclosing isosceles triangle with fixed apex angle α.

The reduction is as follows. Let $Z = \{z_1, \ldots, z_n\}$ be an instance of the Max-Gap problem for points on the first quadrant of the unit circle centered at the origin of the coordinates system, where $z_i = (x_i, y_i)$, for $i = 1, \ldots, n$. In $O(n)$ time we can compute the first, the second, the penultimate, and the last point of Z in the x-coordinate order; without loss of generality we can assume that z_1, z_2, z_{n-1}, and z_n are these points of Z, respectively. Furthermore, we can assume that $z_1 = (0, 1)$.

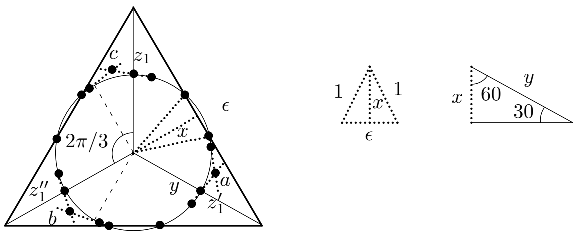

Figure 10. Lower bound construction for the enclosing isosceles triangle with given height and minimum apex angle.

Make two copies of Z on the unit circle by rotating clockwise the points of Z by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, respectively, as it is shown in Figure 10. Let $Z' = \{z'_1, \ldots, z'_n\}$ and $Z'' = \{z''_1, \ldots, z''_n\}$ be the points of the two copies. Notice that the rotation of each point can be done in constant time. We put a point a at the intersection point of the line passing through the points z_{n-1} and z_n, and the line passing through the points z'_1 and z'_2. We put a point b at the intersection point of the line passing through the points z'_{n-1} and z'_n, and the line passing through the points z''_1 and z''_2. Finally, we put a point c at the intersection point of the line passing through the points z''_{n-1} and z''_n, and the line passing through the points z_1 and z_2. Now, let P be the set of all these $3n+3$ points, i.e., $P = Z \cup Z' \cup Z'' \cup \{a, b, c\}$. The three points $\{a, b, c\}$ force that each one of the three sides of the minimum (in height) enclosing isosceles triangle for P contains two points of Z, Z' and Z'' since we can always rotate the isosceles triangle decreasing its height till each side share two points of P.

Now, by construction, the minimum (in height) enclosing isosceles triangle for P has to be an equilateral triangle such that each of the sides pass through two consecutive points of P. In fact, the enclosing equilateral triangle can be thought as a caliper with three sides.

Because the problem of the minimum height for equilateral triangle is equivalent to minimum perimeter problem, each side of the minimum enclosing equilateral triangle has to contain two consecutive points of Z, of Z', and of Z'' defining the maximum gap for Z, Z', and Z''. Since the Max-Gap problem for points in the first quadrant on the unit circle requires $\Omega(n \log n)$ operations in the algebraic computation tree model [13,18], the theorem follows. Recall Observation 1 for minimum area or perimeter. □

Theorem 6. *Given P and a height h, computing the enclosing isosceles triangle with height h containing P with minimum apex angle α, $0 < \alpha < \pi$, minimum perimeter, and minimum area requires $\Omega(n \log n)$ operations in the algebraic computation tree model.*

Proof. We use the same construction as in Theorem 5, therefore we use the same set of points $P = Z \cup Z' \cup Z'' \cup \{a, b, c\}$ but using a reduction to the problem of deciding whether the Max-Gap(Z) is greater than or equal to a given positive value ϵ, which is part of the input. Thus, given an instance Z and $\epsilon > 0$, let $h = \frac{3}{2}\sqrt{4-\epsilon^2}$. This formula can be obtained from the triangles in Figure 10 since

$$x = \frac{\sqrt{4-\epsilon^2}}{2}, \quad y = \sqrt{4-\epsilon^2}, \quad h = x + y = \frac{3\sqrt{4-\epsilon^2}}{2}.$$

Now we compute the enclosing isosceles triangle for P with height h and minimum apex angle. By construction, it is clear that if Max-Gap(Z)$< \epsilon$, then any enclosing isosceles (equilateral) triangle has height $> h$, i.e., there exists no solution. On the other hand, if Max-Gap(Z)$\geq \epsilon$, then there exists an enclosing isosceles (equilateral) triangle with height h which three contact points of $CH(P)$ are one of the extremes of an interval defining a gap $\geq \epsilon$ in Z.

Since the Max-Gap(Z)$\geq \epsilon$ problem for a point set Z in the first quadrant on the unit circle requires $\Omega(n \log n)$ operations in the algebraic computation tree model [1], the theorem follows. Recall Observation 2 for minimum area or perimeter. □

3. α-cones

In this section we consider a three dimensional variant of the problems defined in the introduction. Let P be a set of n points in $\mathbb{R}^3$. We assume that the points are in general position. Recall, that the enclosing structure is a cone with angle α at the apex and a circular base (Figure 11). We will call this an α-cone. We wish to compute an enclosing α-cone whose base is contained in a horizontal plane with minimum height. This is the 3-dimensional problem equivalent to the 2-dimensional problem addressed in Subsection 2.1. In the optimal solution, excluding the base of the cone, only the vertices of the upper hull of $CH(P)$ can be in contact with the cone. We proceed as follows.

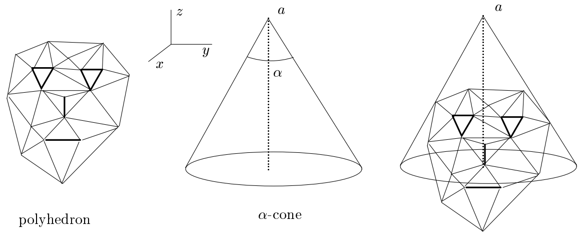

Figure 11. Putting an α-cone to a convex polyhedron.

(1) Let H be a horizontal plane such that P is above H. For each $p_i \in P$ compute a vertical α-cone with apex p_i. Compute the intersection of these α-cones with H, resulting in n circles with different radii (Figure 12).
(2) Compute the smallest enclosing circle C' for this set of circles. C' has at least three (in some cases two) contact points. Knowing the center and radius of C', we can compute the apex of the vertical α-cone of minimum height. Computing the smallest enclosing circle of a set of circles can be done in linear time [14].

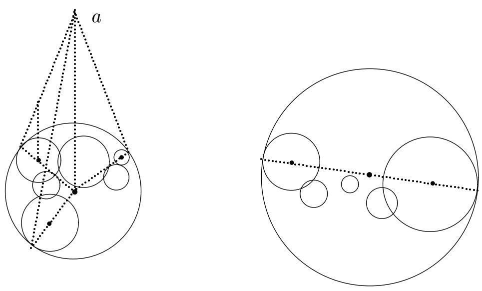

Figure 12. Computing a vertical α-cone.

In the 0-cone the apex goes to infinity and thus, the 0-cone can be interpreted as a cylinder, i.e., the smallest enclosing vertical cylinder, which can be computed as follows: project the points on the plane H and compute the center c and the radius r of the smallest enclosing circle of the projecting points in linear time. The axis of the smallest enclosing vertical cylinder is the vertical line passing through c and the radius is r.

The π-cone can be interpreted as a horizontal plane passing through the point of P with the biggest z-coordinate, which can be computed in linear time. Notice that for any given orientation there always exists an α-cone in that orientation. The α-cone with minimum height has at least one point on its base. We conclude with the following.

Theorem 7. *Given P and α, $0 \leq \alpha \leq \pi$, the α-cone enclosing P with horizontal base and minimum height can be computed in linear time.*

The problem of computing the α-cone enclosing P of minimum height with horizontal base and the problem of computing the smallest enclosing circle for a set of n circles in 2D are equivalent problems in the following sense:

- An instance of the first problem for a point set in 3-dimension can be converted in an instance of the smallest enclosing circle just following the steps above.
- On the other hand, given an instance of the smallest enclosing circle, i.e., a set of n circles in a plane H, then we do the following.

  (1) For each circle C_i with center $c_i = (a_i, b_i)$ and radius r_i, compute the point (above H) located in a point p_i on the vertical line passing through the center c_i and such that the angle formed by the segment $p_i c_i$ and the radius r_i is α (Figure 12).

  (2) In this way we obtain a set of n points $P = \{p_1, \ldots, p_n\}$ in 3-dimension such that if we compute the α-cone enclosing P with horizontal base and minimum height and project its apex on the plane H containing the circles, then the projected point is the center of the smallest enclosing circle for the set of circles.

3.1. *Problem 1: PTAS for minimum height*

We describe a Polynomial Time Approximation Scheme for Problem 1, which is an extension of the 2-dimension algorithm. Given P and α, $0 < \alpha < \pi$, the algorithm finds an enclosing α-cone with apex angle α whose height is within a multiplicative factor $(1+\epsilon)$ of the optimal height. The running time is $O(n)$ with a constant that depends on α. As α tends to 0 or π, the running time tends to infinity. We describe the algorithm below.

3-dimension-PTAS-algorithm

(1) Choose a set S of orientations on the unit sphere so that any orientation d on the unit sphere has some orientation $d^* \in S$ where the angle between d and d^* is less than δ, with

$$\sin\delta = \frac{\epsilon \sin\alpha}{2 - \cos\alpha}.$$

This can be accomplished using $O(\delta^{-2})$ orientations as follows. Consider a system of longitudes and latitudes on the unit sphere such that successive longitudes and latitudes differ by an angle of δ. Consider S to be the set of orientations defined by the intersections of these longitudes and latitudes.

(2) For each orientation in S, compute the enclosing α-cone of minimum height using Theorem 7. Return the enclosing α-cone of minimum height among all computed enclosing α-cones.

To prove the correctness of this algorithm we will need the following lemma.

Lemma 6. *Let C be an α-cone with apex angle α and height h. Let C' be the α-cone of minimum height h' containing C, with apex angle α, and circular base tilted by an angle of δ. Then, we have that*

$$h' = h\left(\frac{(2-\cos\alpha)\sin\delta}{\sin\alpha} + \cos\delta\right).$$

Proof. Figure 13 shows an α-cone C of height h inside another α-cone C' tilted by an angle of δ radians. The only thing we have to prove is that the axes lines ℓ_1 and ℓ_2 of the two α-cones intersect in a common point. Then the intersection of the two cones with the plane defined by the lines ℓ_1 and ℓ_2 produces two tilted isosceles triangles T and T' as in Figure 2 for the 2 dimensional case. Therefore applying the same reasoning we can calculate the height of the outer α-cone to be

$$h' = h\left(\frac{(2-\cos\alpha)\sin\delta}{\sin\alpha} + \cos\delta\right).$$

In order to prove that the lines ℓ_1 and ℓ_2 intersect in a common point, we do the following. Let H be the plane containing the base circle of the α-cone C. The intersection between plane H and the α-cone C' is an ellipse which contains the base circle of the α-cone C (see Figure 13). The circle is tangent to the ellipse in two points A, B, which are symmetric with respect to the large axis of the ellipse, $\overline{EF}$. This fact can be proved by taking the normals of the supporting lines of the circle in this two points and showing that its intersection point is in the large axis $\overline{EF}$ of the ellipse because otherwise we get a contradiction. As a consequence, we get that the axes lines ℓ_1 and ℓ_2 of the two α-cones intersect in a common point and define a plane which is the plane perpendicular to the segment $\overline{AB}$ and contains the large axis $\overline{EF}$ of the ellipse. □

Theorem 8. *The 3-dimension algorithm returns an enclosing α-cone whose height is within a multiplicative factor $(1+\epsilon)$ of the optimal height. The running time of the algorithm is $O(n)$ for any fixed α, such that $0 < \alpha < \pi$.*

Proof. Same as for Theorem 1 except that the 3-dimension algorithm applies Theorem 7 $O(\delta^{-2})$ times. Consider the axis orientation d of the optimal α-cone C. At least one orientation d' of set S will be within δ of d, where δ is as defined in

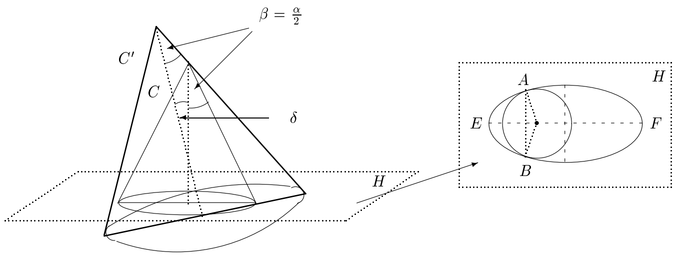

Figure 13. α-cone C inside the α-cone C' tilted by an angle of δ.

Lemma 6. The optimal α-cone C' over C with axis along d' will also be an α-cone for the given point set. Hence the height of C' is an upper bound on the output of the algorithm. From Lemma 6 and our choice of δ this height is less than or equal to $h(\epsilon + \cos\delta)$ which is less than or equal to $h(\epsilon + 1)$.

Thus the algorithm solves the simpler subproblem $O(\delta^{-2})$ times using using Theorem 7. Since δ^{-2} is constant for $0 < \alpha < \pi$, the running time of the algorithm is $O(n)$. □

We conclude with the following two open questions.

Question 3.1. Given a set P of n points in 3-dimension and an apex angle α, efficiently compute the apex and the direction of the α-cone for P with minimum height.

Notice that for any fixed orientation of the α-cone there always exists a solution. For $\alpha = 0$, the apex of the cone goes to infinity and the problem becomes the computation of the smallest enclosing cylinder for the set P (see [2,7,19] for several works on this problem). For $\alpha = \pi$, the conic body of the cone becomes a plane through a point of P and the minimum height corresponds to the distance of the parallel supporting planes of P which define the width of P (see [9,11] for the problem of computing the width of a set of points in three dimensions).

Notice that the solution of this problem in the two dimensional setting was an α-cloud which can be thought as the upper envelope of the set of at most a linear number of circles defined by the contact points and the value α (Figures 3 and 9). Similarly, the solution for the three dimensional setting is the upper envelope of spherical surfaces. Therefore the problem is reduced to computing all the (at most $O(n^3)$) triples of contact points, the corresponding surfaces, and their upper envelope. Then, the additional step remaining is how to compute the α-cone with minimum height.

Question 3.2. Given a set P of n points in 3-dimension and a height h, efficiently compute the apex and the direction of the α-cone with height h and minimum aperture angle.

We observe the following about the above open problem. Solutions exist only for those orientation $\vec{u}$ such that $w_{\vec{u}}(P) \leq h$. The simpler subproblem to compute a vertical α-cone with a given height h and minimum apex angle α, can be solved in constant time if we know three contact points.

Bibliography

1. E. M. Arkin, F. Hurtado, J. S. B. Mitchell, C. Seara, and S. S. Skiena, Some lower bounds on geometric separability problems, *International Journal of Computational Geometry and Applications*, Vol. **16** No. 1 (2006) pp. 1–26.
2. P. K. Agarwal, B. Aronov, and M. Sharir, Line traversals of balls and smallest enclosing cylinders in 3 dimensions, *8th ACM-SIAM Sympos. Discr. Algorit.* (1997) pp. 483–492.
3. A. Aggarwal and J. Park, Notes on searching in multi-dimensional monotone arrays, *FOCS* (1988) pp. 497–512.
4. B. Bhattacharya and A. Mukhopadhyay, On the minimum perimeter triangle enclosing a convex polygon, *JCDCG* (2002) pp. 84–96.
5. P. Bose, F. Hurtado, E. Omaña-Pulido, J. Snoeyink, and G. T. Toussaint, Aperture angle problems, *Algorithmica*, Vol. **33** (2002) pp. 411–435.
6. B. Becker, P. G. Franciosa, S. Gschwind, S. Leonardi, T. Ohler, and P. Widmayer, Enclosing a set of object by two minimum area rectangles, *J. of Algorithms* Vol. **21** (1996) pp. 520–541.
7. T. M. Chan, Approximating the diameter, width, smallest enclosing cylinder, and minimum-width annulus, *International Journal of Computational Geometry and Applications* Vol. **12** (2002) pp. 67–85.
8. J. S. Chang and C. K. Yap, A polynomial solution for potato-peeling and other polygon inclusion and enclosure problems, *FOCS* (1984) pp. 408–416.
9. B. Chazelle, H. Edelsbrunner, L. Guibas, and M. Sharir, Diameter, width, closest line pair, and parametric searching, *8th Ann. Symp. on Comput. Geom.* (1992) pp. 120–129.
10. N. De Pano, Polygon approximation with optimized polygonal enclosures: applications and algorithms, PhD Thesis, (1987).
11. M. E. Houle and G. T. Toussaint, Computing the width of a set, *IEEE Trans. on Pattern Analysis and Machine Intelligence* Vol. **10** No. 5 (1988) pp. 761–765.
12. V. Klee and M. C. Laskowski, Finding the smallest triangles containing a given convex polygon, *J. of Algorithms* Vol. **6** (1985) pp. 359–375.
13. D. T. Lee and Y. F. Wu, Geometric complexity of some location problems, *Algorithmica* Vol. **1** (1986).
14. N. Megiddo, On the ball spanned by balls, *Disc. Comput. Geom.* Vol **4** (1989) pp. 605–610.
15. E. Omaña-Pulido and G. T. Toussaint, Aperture-angle optimization problems in 3 dimensions, *J. of Mathem. Modell. and Algorit.* Vol. **1** (2002) pp. 301–329.
16. F. P. Preparata and M. I. Shamos *Computational Geometry, An Introduction* (Springer, 1988).
17. J. O'Rourke, A. Aggarwal, S. Maddila, and M. Baldwin, An optimal algorithm for finding minimal enclosing triangles, *J. of Algorithms* Vol. **7** (1986).
18. V. Sacristán, Lower bounds for some geometric problems, Technical Report MA-IR-98-0034, Dpt. de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, 1998.
19. E. Schoemer, J. Sellen, M. Teichmann, and C.-K. Yap, Smallest enclosing cylinders, *Algorithmica* Vol. **27** No. 2 (2000) pp. 170–186.

20. M. Teichmann, Shoving a table into a corner, Snapshots of computational and discrete geometry, Technical Report SOCS-88.11, McGill University (1988) pp. 99–118.
21. G. Toussaint, Solving geometric problems with the rotating calipers, *Proceedings of IEEE MELECON'83*, pp. A10.02/1-4 (1983).