Relative position of a point with respect to a circle

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Proposition 1 The intersection of the paraboloid whose equation is \( z = x^2 + y^2 \) with a non vertical plane is a curve that projects orthogonally onto a circle in the plane \( z = 0 \).

Proof: The intersection of the paraboloid \( z = x^2 + y^2 \) with a plane with equation \( z = 2ax + 2by + c \), is a curve whose projection onto \( z = 0 \) has equation \( x^2 + y^2 = 2ax + 2by + c \) i.e., \( (x-a)^2 + (y-b)^2 = c + a^2 + b^2 \). Depending on the value of \( c \), three cases can happen:

- If \( c < -a^2 - b^2 \), the intersection is empty.
- If \( c = -a^2 - b^2 \), the intersection is the point \((a, b, a^2 + b^2)\).
- If \( c > -a^2 - b^2 \), the intersection is a curve (in fact, an ellipse) that projects onto the circle of the plane \( z = 0 \) whose center is \((a, b)\) and whose radius is \( r = \sqrt{c + a^2 + b^2} \). □

Proposition 2 Let \( x, a, b \) and \( c \) be four points in the plane \( z = 0 \), and let \( x^*, a^*, b^* \) and \( c^* \) respectively be their vertical projections onto the paraboloid \( z = x^2 + y^2 \). If \( a, b \) and \( c \) are not aligned, let \( C \) be the circle through \( a, b \) and \( c \), and let \( \pi \) be the plane through \( a^*, b^* \) and \( c^* \).

Then:

- The point \( x \) lies in the circle \( C \) if and only if \( x^* \) lies in the plane \( \pi \).
- The point \( x \) lies in the interior of the circle \( C \) if and only if \( x^* \) lies in the lower half-space determined by \( \pi \).
- The point \( x \) lies in the exterior of the circle \( C \) if and only if \( x^* \) lies in the upper half-space determined by \( \pi \). □

Proof: From the previous proposition we know that the plane \( \pi \) intersects the paraboloid in a curve that projects onto \( C \). Moreover, due to the convexity of the paraboloid, the points of the paraboloid which lie below \( \pi \) are projected onto points interior to \( C \), and those which lie above \( \pi \) are projected onto point external to \( C \). □
Proposition 3 Let $x = (x_1, x_2, x_3)$, $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and $c = (c_1, c_2, c_3)$ be four points in space. If $a$, $b$ and $c$ are not aligned, let $\pi$ be the plane that they define. If $\det(x, a, b, c)$ is the following determinant:

$$
\det(x, a, b, c) = \begin{vmatrix}
  x_1 & a_1 & b_1 & c_1 \\
  x_2 & a_2 & b_2 & c_2 \\
  x_3 & a_3 & b_3 & c_3 \\
  1 & 1 & 1 & 1
\end{vmatrix},
$$

then:

- The point $x$ belongs to the plane $\pi$ if and only if $\det(x, a, b, c) = 0$.
- The point $x$ belongs to the “positive” half-plane defined by $\pi$ if and only if $\det(x, a, b, c) > 0$.
- The point $x$ belongs to the “negative” half-plane defined by $\pi$ if and only if $\det(x, a, b, c) < 0$.

(The concepts “positive” and “negative” follow the right hand rule.)

Proof: The situation is similar to the 2D case. In the 3D case, $\frac{1}{6} \det(x, a, b, c)$ equals the oriented volume of the tetrahedron with vertices $a$, $b$, $c$ and $x$, where the orientation is defined by the three sorted vectors $\overrightarrow{ab}$, $\overrightarrow{ac}$ and $\overrightarrow{ax}$, according to the right hand rule. □

Corollary 4 Let $a$, $b$ and $c$ be three non aligned points in the plane, that appear angularly sorted in counterclockwise order in the circle $C$ that they determine. Let $x$ be any point in the plane.

Then:

- The point $x$ lies in the circle $C$ if and only if $\det(x^*, a^*, b^*, c^*) = 0$.
- The point $x$ lies in the interior of $C$ if and only if $\det(x^*, a^*, b^*, c^*) < 0$.
- The point $x$ lies in the exterior of $C$ if and only if $\det(x^*, a^*, b^*, c^*) > 0$.

Observation 5 To compute the determinant of the previous corollary, it is convenient to do the calculations in terms of the differences between the values of the coordinates of the points involved, and to avoid making calculations (specially, products) in terms of the coordinate values, if possible. In order to do that, the following identity can be used:

$$(x_1, x_2, x_3) - (a_1, a_2, a_3) = (b_1, b_2, b_3) - (c_1, c_2, c_3)$$