# Some fractional functional inequalities

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# Fractional filtration equation

$$\begin{cases} u_t + (-\Delta)^{\sigma/2}(\varphi(u)) = 0, & x \in \Omega,\ t > 0 \\ u(\cdot, 0) = f \in L^1(\Omega) \end{cases}$$

$$0 < \sigma < 2$$

- DOMAIN: $\Omega = \mathbb{R}^N$ *or* $\Omega$ bounded
- BOUNDARY DATA ($\Omega$ BOUNDED): $u = 0, \quad x \in \partial\Omega, t > 0$
- NON-LINEARITIES: $\varphi$ continuous, non-decreasing, $\varphi(0) = 0$
  - Examples: $\begin{cases} \varphi(u) = |u|^{m-1}u, & m > 0 \\ \varphi(u) = \log(1+u), & f \geq 0 \end{cases}$

$A$ non-negative, self-adjoint, linear operator

$$A^\alpha u(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left(e^{-tA} u(x) - u(x)\right) \frac{dt}{t^{1+\alpha}}$$

- $\lambda^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left(e^{-t\lambda} - 1\right) \frac{dt}{t^{1+\alpha}}, \qquad \lambda > 0$

$$(-\Delta)^{\sigma/2} u(x) = \frac{1}{\Gamma(-\frac{\sigma}{2})} \int_0^\infty \left(e^{t\Delta} u(x) - u(x)\right) \frac{dt}{t^{1+\frac{\sigma}{2}}}$$

# Non-local operator

$\Omega = \mathbb{R}^N$

- $\mathcal{F}\big((-\Delta)^{\sigma/2}u\big)(\xi) = |\xi|^\sigma\, \mathcal{F}(u)(\xi)$
- $(-\Delta)^{\sigma/2}u(x) = C_{N,\sigma}\text{P.V.} \displaystyle\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+\sigma}}\, dy$

# Non-local operator

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$\Omega$ bounded, homogeneous Dirichlet B. C.

- $u = \sum_{k=1}^{\infty} u_k\phi_k \qquad \Rightarrow \qquad (-\Delta)^{\sigma/2}u = \sum_{k=1}^{\infty} \lambda_k^{\sigma/2} u_k \phi_k$

$$\begin{cases} -\Delta\phi_k = \lambda_k\phi_k, & x \in \Omega, \\ \phi_k = 0, & x \in \partial\Omega \end{cases}$$

# Motivation

- Non-linear generalization of the *fractional heat equation*

$$u_t + (-\Delta)^{\sigma/2} u = 0$$

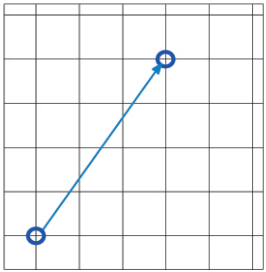
- Non-local generalization of the *filtration equation*

$$u_t - \Delta(\varphi(u)) = 0$$

(Other possible generalizations [Caffarelli-Vázquez, 2009])

- Hydrodynamic limit of zero range processes [Jara, 2009]

## A jump process



- $u(x,t)$: probability of lying at $x \in h\mathbb{Z}^N$ at time $t \in \tau\mathbb{Z}$.
- $\mathcal{K}(k-j)$: probability of jumping from $hk$ to $hj$ in a time unit $\tau$.
- $u(x,t+\tau) = \sum_{k\in\mathbb{Z}^N} \mathcal{K}(k)u(x+hk,t)$

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \sum_{k\in\mathbb{Z}^N} \frac{\mathcal{K}(k)}{\tau}\big(u(x+hk,t)-u(x,t)\big)$$

- Jumps only to closest neighbours, $\tau = h^2 \to 0 \quad \Rightarrow \quad$ Heat equation

## Random walk with long jumps

- $\mathcal{K}(z) = C_{N,\sigma}|z|^{-(N+\sigma)}, \qquad \sigma \in (0,2)$
- $\tau = h^\sigma$: $\qquad \dfrac{\mathcal{K}(k)}{\tau} = h^N \mathcal{K}(hk)$
- $\dfrac{u(x,t+\tau) - u(x,t)}{\tau} = h^N \sum_{k\in\mathbb{Z}^N} \mathcal{K}(hk)\big(u(x+hk) - u(x)\big)$

$$\tau = h^\sigma \to 0$$

$$\begin{aligned} u_t &= \text{P.V.} \int_{\mathbb{R}^N} \mathcal{K}(z)\big(u(x+z) - u(x)\big)\,dz \\ &= \underbrace{C_{N,\sigma}\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x+z) - u(x)}{|z|^{N+\sigma}}\,dz}_{-(-\Delta)^{\sigma/2}u(x)} \end{aligned}$$

# Mild solution

- THEOREM [Crandall-Pierre, 1982]: $A$ linear operator such that

$$\exists \text{ \textit{mild} solution (ITD): } \begin{cases} u_t + Au = 0, \\ u(0) = f \in L^1 \end{cases}$$

$$+$$

conditions on $\varphi$

$$\Downarrow$$

$$\exists \text{ mild solution: } \begin{cases} u_t + A\varphi(u) = 0, \\ u(0) = f \in L^1 \end{cases}$$

- Abstract construction $\Rightarrow$
  - Not enough information to prove that mild $\Rightarrow$ weak.
  - No estimates $\Rightarrow$ No further properties.

# Integration by parts / Functional space

$$\int_{\mathbb{R}^N} (-\Delta)^{\sigma/2}\varphi\,\psi = \int_{\mathbb{R}^N} |\xi|^{\sigma}\hat{\varphi}\,\hat{\psi} = \int_{\mathbb{R}^N} |\xi|^{\sigma/2}\hat{\varphi}\,|\xi|^{\sigma/2}\hat{\psi}$$

$$= \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4}\varphi\,(-\Delta)^{\sigma/4}\psi$$

# Integration by parts / Functional space

- $$\int_{\mathbb{R}^N} (-\Delta)^{\sigma/2}\varphi\,\psi = \int_{\mathbb{R}^N} |\xi|^{\sigma}\hat{\varphi}\,\hat{\psi} = \int_{\mathbb{R}^N} |\xi|^{\sigma/2}\hat{\varphi}\,|\xi|^{\sigma/2}\hat{\psi} = \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4}\varphi\,(-\Delta)^{\sigma/4}\psi$$

- $$\|\varphi\|_{\dot{H}^{\sigma/2}} = \left(\int_{\mathbb{R}^N} |\xi|^{\sigma}|\hat{\varphi}|^2\,d\xi\right)^{1/2} = \|(-\Delta)^{\sigma/4}\varphi\|_2 = \left(\int_{\mathbb{R}^N}\left(C_{N,\sigma}\,\text{P.V.}\int_{\mathbb{R}^N}\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+\frac{\sigma}{2}}}\,dy\right)^2 dx\right)^{1/2}$$

## Weak ($L^1$-energy) solution

- $u \in C([0,\infty) : L^1(\mathbb{R}^N))$, $\varphi(u) \in L^2_{\text{loc}}((0,\infty) : \dot{H}^{\sigma/2}(\mathbb{R}^N))$

- $\displaystyle\int_0^\infty \int_{\mathbb{R}^N} u \frac{\partial \psi}{\partial t}\, dxds - \int_0^\infty \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4}(\varphi(u))(-\Delta)^{\sigma/4}\psi\, dxds = 0$,
  $\forall \psi \in C^1_{\mathsf{c}}(\mathbb{R}^N \times (0,\infty))$

- $u(\cdot,0) = f$ a.e.

# Weak/strong solutions

## Weak ($L^1$-energy) solution

- $u \in C([0,\infty) : L^1(\mathbb{R}^N))$, $\varphi(u) \in L^2_{\mathrm{loc}}((0,\infty) : \dot{H}^{\sigma/2}(\mathbb{R}^N))$
- $$\int_0^\infty \int_{\mathbb{R}^N} u \frac{\partial \psi}{\partial t}\, dxds - \int_0^\infty \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4}(\varphi(u))(-\Delta)^{\sigma/4}\psi\, dxds = 0,$$
  $\forall \psi \in C^1_{\mathrm{c}}(\mathbb{R}^N \times (0,\infty))$
- $u(\cdot,0) = f$ a.e.

## Strong solution

- Weak ($L^1$-energy) solution
- $\partial_t u \in L^1_{\mathrm{loc}}(\mathbb{R}^N \times (0,\infty))$ + ...

# Difficulties

- $(-\Delta)^{\sigma/4}(\varphi\psi) = ?$
- $(-\Delta)^{\sigma/4}(\varphi \circ \psi) = ?$
- $(-\Delta)^{\sigma/4}\psi$ not compactly supported even when $\psi$ is
  - $(-\Delta)^{\sigma/4}$ *non-local* operator

## $\sigma$-harmonic extension [Caffarelli-Silvestre, 2007]

$$v = \mathrm{E}(g) : \quad \begin{cases} L_\sigma v \equiv \operatorname{div}(y^{1-\sigma}\nabla v) = 0, & \mathbb{R}^{N+1}_+ = \{x \in \mathbb{R}^N,\, y > 0\} \\ v(x,0) = g(x), & x \in \mathbb{R}^N \end{cases}$$

- $v(x,y) = \int_{\mathbb{R}^N} P(x-\xi, y) g(\xi)\, d\xi, \qquad P(x,y) = d_{N,\sigma} \dfrac{y^\sigma}{(|x|^2 + |y|^2)^{(N+\sigma)/2}}$
- $$\lim_{y\to 0^+} y^{1-\sigma}\frac{\partial v}{\partial y} = \sigma \lim_{y\to 0^+} \frac{v(x,y) - v(x,0)}{y^\sigma}$$

$$= \sigma \lim_{y\to 0^+} \int_{\mathbb{R}^N} \frac{P(x-\xi, y)}{y^\sigma}(g(\xi) - g(x))\, d\xi = -\frac{\sigma d_{N,\sigma}}{C_{N,\sigma}}(-\Delta)^{\sigma/2} g$$

$$\frac{\partial v}{\partial y^\sigma} \equiv \mu_\sigma \lim_{y\to 0^+} y^{1-\sigma}\frac{\partial v}{\partial y} = -(-\Delta)^{\sigma/2} g$$

# An equivalent local problem

- $w = \mathrm{E}(\varphi(u)), \quad u = \varphi^{-1}(\mathrm{Tr}(w))$

$$\begin{cases} L_\sigma w = 0, & (x,y) \in \mathbb{R}^{N+1}_+,\ t > 0, \\ \dfrac{\partial w}{\partial y^\sigma} - \dfrac{\partial \varphi^{-1}(w)}{\partial t} = 0, & x \in \mathbb{R}^N, y = 0,\ t > 0, \\ w = \varphi(f), & x \in \mathbb{R}^N, y = 0,\ t = 0. \end{cases}$$

- Some proofs (e.g. existence) may be easier in this formulation!
- Dynamical boundary conditions (Amann, Escher, Fila, Vitillaro, ...)

# Weak solution (local problem)

## Weak ($L^1$-energy) solution

- $u \in C([0,\infty) : L^1(\mathbb{R}^N))$, $w \in L^2_{\text{loc}}((0,\infty) : X^\sigma(\mathbb{R}^{N+1}_+))$;
- $\displaystyle\int_0^\infty \int_{\mathbb{R}^N} u \frac{\partial \psi}{\partial t}\, dxds - \mu_\sigma \int_0^\infty \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \langle \nabla w, \nabla \psi \rangle\, dxdyds = 0,$
  $\forall \psi \in C^1_{\mathsf{c}}\left(\overline{\mathbb{R}^{N+1}_+} \times (0,\infty)\right)$;
- $u(\cdot, 0) = f$ a.e.

► $\displaystyle\|v\|_{X^\sigma} = \left(\mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} |\nabla v|^2\, dxdy\right)^{1/2}$

# Extension / trace operators

- $\mathrm{E} : \dot{H}^{\sigma/2}(\mathbb{R}^N) \to X^\sigma(\mathbb{R}^{N+1}_+)$ isometry [Caffarelli-Silvestre, 2007]

- $$\mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \langle \nabla \mathrm{E}(\varphi), \nabla \mathrm{E}(\psi) \rangle = \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4} \varphi \, (-\Delta)^{\sigma/4} \psi$$

- $$\mathrm{Tr}(\Psi_1) = \mathrm{Tr}(\Psi_2)$$
$$\Downarrow$$
$$\int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \langle \nabla \mathrm{E}(\varphi), \nabla \Psi_1 \rangle = \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \langle \nabla \mathrm{E}(\varphi), \nabla \Psi_2 \rangle$$

- $\mathrm{Tr} : X^\sigma(\mathbb{R}^{N+1}_+) \to \dot{H}^{\sigma/2}(\mathbb{R}^N)$ surjective and continuous

- TRACE EMBEDDING: $\|\Phi\|_{X^\sigma} \geq \| \mathrm{E}(\mathrm{Tr}(\Phi))\|_{X^\sigma} = \| \mathrm{Tr}(\Phi)\|_{\dot{H}^{\sigma/2}}$

# Stroock-Varopoulos inequality, $\sigma = 2$

- $\psi' = (\Psi')^2$:

$$
\begin{aligned}
\int_{\mathbb{R}^N} \psi(v)(-\Delta)v &= \int_{\mathbb{R}^N} \langle \nabla\psi(v), \nabla v\rangle \\
&= \int_{\mathbb{R}^N} \psi' \quad |\nabla v|^2 \\
&= \int_{\mathbb{R}^N} |\nabla \Psi(v)|^2 \\
&= \int_{\mathbb{R}^N} \left|(-\Delta)^{1/2}\Psi(v)\right|^2
\end{aligned}
$$

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## Stroock-Varopoulos inequality, $0 < \sigma < 2$ [S, 1984], [V, 1985]

THM: $\psi' = (\Psi')^2 \quad \Rightarrow \quad \int_{\mathbb{R}^N} \psi(v)(-\Delta)^{\sigma/2} v \geq \int_{\mathbb{R}^N} \left|(-\Delta)^{\sigma/4}\Psi(v)\right|^2$

$$
\begin{aligned}
\int_{\mathbb{R}^N} \psi(v)(-\Delta)^{\sigma/2} v &= \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4}\psi(v)(-\Delta)^{\sigma/4} v \\
&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \langle \nabla \mathrm{E}(\psi(v)), \nabla \mathrm{E}(v)\rangle \\
&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}\ \psi' \quad \langle \nabla \mathrm{E}(v), \nabla \mathrm{E}(v)\rangle \\
&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \left|\nabla \Psi(\mathrm{E}(v))\right|^2 \\
&\geq \int_{\mathbb{R}^N} \left|(-\Delta)^{\sigma/4}\Psi(v)\right|^2
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&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}\ \psi' \quad \langle \nabla \mathrm{E}(v), \nabla \mathrm{E}(v)\rangle \\
&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}\ |\nabla \Psi(\mathrm{E}(v))|^2 \\
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&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}\ \psi' \quad \langle \nabla \mathrm{E}(v), \nabla \mathrm{E}(v)\rangle \\
&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \left|\nabla \Psi(\mathrm{E}(v))\right|^2 \\
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&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} (\Psi')^2 \langle \nabla \mathrm{E}(v), \nabla \mathrm{E}(v) \rangle \\
&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \left| \nabla \Psi(\mathrm{E}(v)) \right|^2 \\
&\geq \int_{\mathbb{R}^N} \left| (-\Delta)^{\sigma/4} \Psi(v) \right|^2
\end{aligned}
$$

## A pointwise inequality [Córdoba-Córdoba, 2003]

- *Local case*: $\phi''(u) \geq 0$.

$$\bullet\ (-\Delta)(\phi(u)) = -\phi''(u)\,|\nabla u|^2 + \phi'(u)(-\Delta)u \leq \phi'(u)(-\Delta)u$$

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$$\bullet\ (-\Delta)(\phi(u)) = -\phi''(u)\,|\nabla u|^2 + \phi'(u)(-\Delta)u \leq \phi'(u)(-\Delta)u$$

- *Non-local case*: $\phi$ convex

$$\bullet\ \phi(u(y)) \geq \phi(u(x)) + \phi'(u(x))(u(y) - u(x))$$

$$\bullet\ (-\Delta)^{\sigma/2}(\phi(u))(x) = C_{N,\sigma}\text{P.V.}\int_{\mathbb{R}^N} \frac{\phi(u(x)) - \phi(u(y))}{|x-y|^{N+\sigma}}\,dy \leq \phi'(u(x))C_{N,\sigma}\text{P.V.}\int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+\sigma}}\,dy = \phi'(u(x))(-\Delta)^{\sigma/2}u(x)$$

# Hardy-Littlewood-Sobolev's inequality

THEOREM [H-L, 1928], [S, 1938]: $0 < \sigma < \min\{2, N\}$

$$\Downarrow$$

$$\|\phi\|_{\frac{2N}{N-\sigma}} \leq C\|(-\Delta)^{\sigma/4}\phi\|_2, \qquad (\dot{H}^{\sigma/2}(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-\sigma}}(\mathbb{R}^N))$$

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- *Riesz potential:* $0 < \sigma < N$,

$$\left((-\Delta)^{-\sigma/2} g\right)(x) = C\int_{\mathbb{R}^N} \frac{g(y)\,dy}{|x-y|^{n-\sigma}}$$

- *HLS inequality:* $1 < p < \infty$, $\quad \dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\sigma}{N}$

$$\|(-\Delta)^{-\sigma/2} g\|_q \le A_{p,q}\|g\|_p$$

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$$1 = N \leq \sigma < 2?$$

- *Bessel potentials* [Aronszajn-Smith, 1961], [Calderon, 1961]:
  - $J_\alpha(g) = (I-\Delta)^{-\alpha/2} g$
  - $J_\alpha(g) = G_\alpha * g, \qquad \widehat{G_\alpha}(\xi) = (1+|\xi|^2)^{-\alpha/2}$
  - $\|G_\alpha\|_1 = 1, \qquad \|J_\alpha(g)\|_p \le \|g\|_p$

- *Bessel potential spaces*: $\mathcal{L}^p_\alpha(\mathbb{R}^N) = J_\alpha(L^p(\mathbb{R}^N))$
  - $\phi = J_\alpha(g) \to \|\phi\|_{\mathcal{L}^p_\alpha} = \|g\|_p$

- THEOREM [Stein, 1961]:
  - $\phi \in \mathcal{L}^p_\alpha(\mathbb{R}^N) \Leftrightarrow \phi, (-\Delta)^{\alpha/2}\phi \in L^p(\mathbb{R}^N)$
  - $\|\phi\|_{\mathcal{L}^p_\alpha} \sim \|\phi\|_p + \|(-\Delta)^{\alpha/2}\phi\|_p$

# Morrey-Sobolev imbedding

- $$H^\alpha(\mathbb{R}^N) = \mathcal{L}^2_\alpha(\mathbb{R}^N) = \{\phi \in L^2(\mathbb{R}^N) : (1+|\xi|^2)^{\alpha/2}\,\hat\phi \in L^2(\mathbb{R}^N)\}$$

THEOREM: $2 > \sigma > N = 1, \quad \phi \in H^{\sigma/2}(\mathbb{R}^N)$

$$\Downarrow$$

$$\phi \in C_0(\mathbb{R}^N), \qquad \|\phi\|_\infty \le C\|\phi\|_{H^{\sigma/2}(\mathbb{R}^N)}$$

- $$\int_{\mathbb{R}^N} |\hat\phi(\xi)|\,d\xi = \int_{\mathbb{R}^N} \frac{(1+|\xi|^2)^{\sigma/4}\,|\hat\phi(\xi)|}{(1+|\xi|^2)^{\sigma/4}}\,d\xi \le \underbrace{\left(\int_{\mathbb{R}^N} \frac{d\xi}{(1+|\xi|^2)^{\frac{\sigma}{2}}}\right)^{\frac12}}_{C} \underbrace{\left(\int_{\mathbb{R}^N} (1+|\xi|^2)^{\frac{\sigma}{2}}\,|\hat\phi(\xi)|^2\,d\xi\right)^{\frac12}}_{\|\phi\|_{H^{\sigma/2}(\mathbb{R}^N)}}$$

# Classical Nash inequality [Nash, 1958]

THEOREM: $0 < \sigma \le 2, \quad \phi \in L^1(\mathbb{R}^N) \cap \dot{H}^{\sigma/2}(\mathbb{R}^N)$

$$\Downarrow$$

$$\|\phi\|_2^{1+\frac{\sigma}{N}} \le C \|\phi\|_1^{\sigma/N} \|\phi\|_{\dot{H}^{\sigma/2}}$$

- $\displaystyle\int_{|\xi|\ge\rho} |\hat{\phi}(\xi)|^2\,d\xi \le \rho^{-\sigma} \int_{|\xi|\ge\rho} |\xi|^\sigma |\hat{\phi}(\xi)|^2\,d\xi \le \rho^{-\sigma}\|\phi\|^2_{\dot{H}^{\sigma/2}}$

- $\displaystyle\int_{|\xi|\le\rho} |\hat{\phi}(\xi)|^2\,d\xi \le \omega_N \rho^N \|\phi\|_1^2$

- $\|\phi\|_2^2 = \|\hat{\phi}\|_2^2 \le \rho^{-\sigma}\|\phi\|^2_{\dot{H}^{\sigma/2}} + \omega_N \rho^N \|\phi\|_1^2$

# A new Nash type inequality [dPQRV, to appear]

THEOREM: $0 < \sigma < 2, \quad \phi \in L^p(\mathbb{R}^N) \cap \dot{H}^{\sigma/2}(\mathbb{R}^N)$

$$\Downarrow$$

$$\|\phi\|_{\frac{N(p+2)}{2N-\sigma}}^{1+\frac{p}{2}} \leq C\|\phi\|_p^{p/2} \, \|\phi\|_{\dot{H}^{\sigma/2}}$$

- Stroock-Varopoulos: $\displaystyle\int_{\mathbb{R}^N} \phi^{\frac{p}{2}}(-\Delta)^{\frac{\sigma}{4}}\phi \geq C\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\sigma}{8}}\phi^{\frac{p+2}{4}}\right|^2$
- Hardy-Littlewood-Sobolev: $\displaystyle\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\sigma}{8}}\phi^{\frac{p+2}{4}}\right|^2 \geq \left(\int_{\mathbb{R}^N} \phi^{\frac{N(p+2)}{2N-\sigma}}\right)^{\frac{2N-\sigma}{2N}}$
- Hölder: $\displaystyle\int_{\mathbb{R}^N} \phi^{\frac{p}{2}}(-\Delta)^{\frac{\sigma}{4}}\phi \leq \|\phi\|_p^{p/2} \, \|\phi\|_{\dot{H}^{\sigma/2}}$

# A Trudinger type inequality [Strichartz, 1972]

THEOREM: $\|\phi\|_{H^{1/2}(\mathbb{R})} \leq 1 \quad \Rightarrow \quad \int_{\mathbb{R}} (e^{\phi^2(x)} - 1)\, dx \leq C$

- Nash:
  - $\|\phi\|_{p+2}^{p+2} \leq c \|(-\Delta)^{1/4}\phi\|_2^2 \, \|\phi\|_p^p$
  - $\|\phi\|_{2k}^{2k} \leq c^{k-1} \|(-\Delta)^{1/4}\phi\|_2^{2(k-1)} \, \|\phi\|_2^2$

- $$\begin{aligned} \int_{\mathbb{R}} (e^{\phi^2(x)} - 1)\, dx &= \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}} \phi^{2k}(x)\, dx \\ &\leq \sum_{k=1}^{\infty} \frac{c^{k-1} \|(-\Delta)^{1/4}\phi\|_2^{2(k-1)}}{(k-1)!} \, \|\phi\|_2^2 \\ &= e^{c\|(-\Delta)^{1/4}\phi\|_2^2} \, \|\phi\|_2^2 \end{aligned}$$

## Gagliardo type norms [Gagliardo, 1957]

- $\phi \in H^{\sigma/2}(\mathbb{R}^N) \Leftrightarrow \phi \in L^2(\mathbb{R}^N), \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+\sigma}}\,dxdy < \infty$

- $\|\phi\|_{H^{\sigma/2}} \sim \|\phi\|_2 + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+\sigma}}\,dxdy$

  - $$\begin{aligned}\int_{\mathbb{R}^N} |\phi(x+h)-\phi(x)|^2\,dx &= \int_{\mathbb{R}^N} |e^{ih\cdot\xi}-1|^2|\hat{\phi}(\xi)|^2\,d\xi \\ &= \int_{\mathbb{R}^N} \sin^2\left(\frac{h\cdot\xi}{2}\right)|\hat{\phi}(\xi)|^2\,d\xi\end{aligned}$$
  - $$\begin{aligned}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+\sigma}}\,dxdy &= \int_{\mathbb{R}^N} |\hat{\phi}(\xi)|^2\left(\int_{\mathbb{R}^N}\frac{\sin^2\left(\frac{h\cdot\xi}{2}\right)}{|h|^{N+\sigma}}\,dh\right)d\xi \\ &= C\int_{\mathbb{R}^N} |\xi|^\sigma|\hat{\phi}(\xi)|^2\,d\xi\end{aligned}$$

- COROLLARY: $\phi \in H^{\sigma/2}(\mathbb{R}^N), \psi \in \mathcal{S}(\mathbb{R}^N) \quad \Rightarrow \quad \phi\psi \in H^{\sigma/2}(\mathbb{R}^N)$

# Besov spaces [Besov, 1961]

- $L^p$ *modulus of continuity*: $\omega_p(t) = \|\phi(x+t) - \phi(x)\|_p$
- $\mathcal{L}^2_\alpha = \left\{ f \in L^2(\mathbb{R}^N) : \left( \int_{\mathbb{R}^N} \frac{(\omega_2(t))^2}{|t|^{N+2\alpha}} \right)^{1/2} < \infty \right\}$
- BESOV SPACES : $\Lambda^{p,q}_\alpha = \left\{ f \in L^p(\mathbb{R}^N) : \left( \int_{\mathbb{R}^N} \frac{(\omega_p(t))^q}{t^{N+\alpha q}} \, dt \right)^{1/q} < \infty \right\}$
- Theorem : $1 < p < \infty, \quad 0 < \alpha < 1$

$$\mathcal{L}^p_\alpha \subset \Lambda^{p,p}_\alpha, \qquad \Lambda^{p,2}_\alpha \subset \mathcal{L}^p_\alpha, \qquad p \geq 2$$

$$\mathcal{L}^p_\alpha \subset \Lambda^{p,2}_\alpha, \qquad \Lambda^{p,p}_\alpha \subset \mathcal{L}^p_\alpha, \qquad p \leq 2$$

# An application: smoothing effect

- $u$ strong solution to $\begin{cases} u_t + (-\Delta)^{\sigma/2}(u^m) = 0, \\ u(\cdot,0) = f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \end{cases}$ , $m > 0$

THEOREM: $p > \max\{1, (1-m)N/\sigma\}$

$$\Downarrow$$

$$\sup_{x\in\mathbb{R}^N} |u(x,t)| \le C\, t^{-N/(N(m-1)+\sigma p)} \|f\|_p^{\sigma p/(N(m-1)+\sigma p)}$$

## Smoothing effect: proof (Moser's iteration)

- Multiply by $u^{p_k-1}$, integrate on $\mathbb{R}^N \times (t_k, t_{k+1})$, $t_k = (1-2^{-k})t$:

$$\int_{\mathbb{R}^N} u^{p_k}(\cdot, t_k) = p_k \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^N} (-\Delta)^{\sigma/2}(u^m) u^{p_k-1} + \int_{\mathbb{R}^N} u^{p_k}(\cdot, t_{k+1})$$

$$\geq C \int_{t_k}^{t_{k+1}} \|(-\Delta)^{\sigma/4} u^{\frac{p_k+m-1}{2}}(\cdot,\tau)\|_2^2\, d\tau \qquad \text{(Stroock-Varopoulos)}$$

$$\geq \frac{C}{\|u(\cdot,t_k)\|_{p_k}^{p_k}} \int_{t_k}^{t_{k+1}} \|u(\cdot,\tau)\|_{p_k}^{p_k} \|(-\Delta)^{\sigma/4} u^{\frac{p_k+m-1}{2}}(\cdot,\tau)\|_2^2\, d\tau \quad (L^p\text{-decay})$$

$$\geq \frac{C}{\|u(\cdot,t_k)\|_{p_k}^{p_k}} \int_{t_k}^{t_{k+1}} \|u(\cdot,\tau)\|_{\frac{N(2p_k+m-1)}{2N-\sigma}}^{2p_k+m-1}\, d\tau \quad \text{(Nash-Gagliardo-Nirenberg)}$$

$$\geq \frac{C2^{-(k+1)}t}{\|u(\cdot,t_k)\|_{p_k}^{p_k}} \|u(\cdot,t_{k+1})\|_{\frac{N(2p_k+m-1)}{2N-\sigma}}^{2p_k+m-1} \qquad (L^p\text{-decay})$$

# Smoothing effect: proof (Moser's iteration)

- $\|u(\cdot,t_{k+1})\|_{\frac{N(2p_k+m-1)}{2N-\sigma}} \le \left(\frac{c}{t}\right)^{\frac{1}{2p_k+m-1}} 2^{\frac{k+1}{2p_k+m-1}} \|u(\cdot,t_k)\|_{p_k}^{\frac{2p_k}{2p_k+m-1}}$

- $p_{k+1} \equiv \dfrac{N(2p_k+m-1)}{2N-\sigma} > p_k$ if $p_0 = p > \dfrac{(1-m)N}{\sigma}$

- Iteration

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