Nonlinear nonlocal diffusion; a transport equation with nonlocal velocity

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The problem

We study the nonlocal logarithmic diffusion problem

$$\begin{cases} \partial_t u + (-\Delta)^{1/2} \log(1+u) = 0 & x \in \mathbb{R},\ t > 0 \\ \\ u(x,0) = f(x) & x \in \mathbb{R} \end{cases}$$

(Joint work with F. Quirós, A. Rodríguez, J.L. Vázquez)

Motivation

- Limit case for the Fractional Porous Medium equation

$$\partial_t u + (-\Delta)^{\sigma/2} u^m = 0$$

in $\mathbb{R}^N$, $N \geq 1$, $0 < \sigma < 2$, $m > 0$.

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- Relation with a transport equation with nonlocal velocity

$$\partial_\tau v - \widetilde{H}(v)\partial_y v = -\partial_y \widetilde{H}(v)$$

where $\widetilde{H}$ is a (nonlinear) nonlocal operator related to the Hilbert transform, [Constantin-Lax-Majda'85, Baker-Li-Morlet'96, Córdoba-Córdoba-Fontelos'05].

Definition (weak solution)

- $u \in C([0,\infty) : L^1(\mathbb{R}))$, $\Phi(u) \in L^2_{\mathrm{loc}}((0,\infty) : \dot{H}^{1/2}(\mathbb{R}))$

$$\int_0^\infty \int_{\mathbb{R}} u\partial_t v \, dxds - \int_0^\infty \int_{\mathbb{R}} (-\Delta)^{1/4}\Phi(u)\,(-\Delta)^{1/4} v \, dxds = 0$$

for every $v \in C_0^1(\mathbb{R} \times (0,\infty))$. $\Phi(u) = \log(1+u)$.

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Definition (strong solution)

- Weak solution
- $\partial_t u \in L^1_{loc}(\mathbb{R}\times(0,\infty))$; (the equation holds a.e.)

Existence

We restrict to nonnegative initial values. (We could also take data $f \geq c > -1$ for most of the results.)

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Consider the following slightly smaller $L\log L$ space

$$\mathcal{X} = \{v \geq 0 \, : \, \int_{\mathbb{R}} (1+v)\log(1+v) < \infty\}$$

The relevant quantity will be the norm $\|\Psi(v)\|_1$, with

$$\Psi(v) = (1+v)\log(1+v) - v, \quad (\Psi' = \Phi)$$

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Theorem

For any $f \in \mathcal{X}$ there exists a unique strong solution u.

- We first prove existence of weak solutions for bounded data (and first in bounded domains), by means of the Crandall-Ligget Theorem (Semigroup Theory).

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- Take $f_n \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $f_n \to f$. Sequence of solutions u_n.
- Contraction plus Semigroup Theory $\Rightarrow\ \exists\, u = \lim_{n\to\infty} u_n$ in $C([0,\infty) : L^1(\mathbb{R}^N))$.

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- Contraction plus Semigroup Theory $\Rightarrow$ $\exists\, u = \lim_{n\to\infty} u_n$ in $C([0,\infty) : L^1(\mathbb{R}^N))$.
- Smoothing effect $\Rightarrow$ u uniformly bounded for $t \geq \tau > 0$.
- Uniform bounds $\Rightarrow$ control of $\Phi(u)$ in $\dot{H}^{1/2}$ $\Rightarrow$ weak $\Rightarrow$ strong.

- $\Phi(u) \in L^2((0,\infty) : H^{1/2}(\mathbb{R}))$, and

$$\int_0^t \int_{\mathbb{R}} |(-\Delta)^{1/4}\Phi(u)|^2(x,s)\,dxds \le \|\Psi(f)\|_1$$

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- $\partial_t u \in L^2((t,T)\times\mathbb{R})$ for every $0 < t < T < \infty$, and

$$\int_t^T \int_{\mathbb{R}} |\partial_t u(x,s)|^2\,dxds \le t^{-1}(1+\|u(\cdot,t)\|_\infty)\|\Psi(f)\|_1$$

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- $\Phi(u) \in L^2((t,\infty) : H^1(\mathbb{R}))$ for every $t>0$, and
$$\int_t^\infty \int_{\mathbb{R}} |\partial_x\Phi(u)|^2(x,s)\,dxds \le t^{-1}(1+\|u(\cdot,t)\|_\infty)\|\Psi(f)\|_1$$

- $\Phi(u)(\cdot,t) \in L^p(\mathbb{R})$ for every $t>0$, and every $1 \le p < \infty$. In fact, Nash-Gagliardo-Nirenberg inequality [dP-Quirós-Rodríguez-Vázquez'11] gives, for $w=\Phi(u)$

$$\|w\|_{p+2}^{p+2} \le c\|(-\Delta)^{1/4} w\|_2^2 \, \|w\|_p^p$$

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valid for every $p\ge 1$. This does not mean $u(\cdot,t)\in L^p(\mathbb{R})$.
- Any L^p norm of u is nonincreasing in time. In fact, for every positive, increasing, convex function g, Stroock-Varopoulos inequality gives

$$\frac{d}{dt}\int_{\mathbb{R}}g(u)=-\int_{\mathbb{R}}(-\Delta)^{1/2}\Phi(u)g'(u)\le-\int_{\mathbb{R}}\left|(-\Delta)^{1/4}G(u)\right|^2\le 0$$

This also includes the case $g=\Psi$, i.e. $\|\Psi(u)(\cdot,t)\|_1\le\|\Psi(f)\|_1$.

Theorem

Assume $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, $p > 1$. Then we have

$$\|u(\cdot,t)\|_\infty \leq \max\{C\, t^{-\frac{1}{p-1}} \|f\|_p^{\frac{p}{p-1}},\ C\, t^{-\frac{1}{p}} \|f\|_p\}$$

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We recall the estimate obtained for the FPME if $m > 0$, $0 < \sigma < 2$, $N \ge 1$, and $\sigma \ge N(1-m)$, which in the case $N = \sigma = 1$ reads:

$$\|u(\cdot,t)\|_\infty \le C\, t^{-\frac{1}{m+p-1}} \|f\|_p^{\frac{p}{m+p-1}}$$

In our case put formally $m = 0$ for u large and $m = 1$ for u small.

Moser iterative technique. Use $v = \frac{u^{r-1}}{r-1} + \frac{u^r}{r}$ as test function in $\mathbb{R} \times (t_1, t_2)$. We get

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$$\begin{aligned} \|u(\cdot,t_1)\|_r^r & +\|u(\cdot,t_1)\|_{r+1}^{r+1} \\ & \geq c\int_{t_1}^{t_2}\int_{\mathbb{R}}(-\Delta)^{1/2}\log(1+u)\Big(\frac{u^{r-1}}{r-1}+\frac{u^r}{r}\Big)\,dxd\tau \\ & \text{Stroock-Varopoulos} \\ & \geq c\int_{t_1}^{t_2}\int_{\mathbb{R}}|(-\Delta)^{1/4}u^{r/2}|^2\,dxd\tau \end{aligned}$$

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Stroock-Varopoulos

$$\geq c \int_{t_1}^{t_2} \int_{\mathbb{R}} |(-\Delta)^{1/4} u^{r/2}|^2 \, dx d\tau$$

We cannot use here Hardy-Littlewood-Sobolev (valid only for $N > \sigma$). Insted, to use Nash-Gagliardo-Nirenberg, we multiply by $\|u(\cdot, \tau)\|_q^q$, to get

Smoothing effect
$L^p \to L^\infty$

$$\|u(\cdot,t_2)\|_{r+q}^{r+q} \le \frac{c}{t_2-t_1}\|u(\cdot,t_1)\|_q^q\Big(\|u(\cdot,t_1)\|_r^r + \|u(\cdot,t_1)\|_{r+1}^{r+1}\Big)$$

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Now the trick. Put $r = r_k = 2^k p$. Integrate in $[t_k, t_{k+1}]$, with $t_k = (1-2^{-k})t$. Consider successively $q = r_k$ and $q = r_k + 1$. Define

$$U_k = \max\{\|u(\cdot,t_k)\|_{r_k},\ \|u(\cdot,t_k)\|_{r_k+1}^{\frac{r_k+1}{r_k}}\}$$

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We obtain

$$U_{k+1} \le (c2^k/t)^{\frac{1}{r_{k+1}}}\, U_k$$

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We obtain

$$U_{k+1} \le (c2^k/t)^{\frac{1}{r_{k+1}}} U_k \le \cdots \le ct^{-\frac{1}{p}} U_0$$

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\|$

$$\|u(\cdot,t)\|_\infty \qquad\qquad ct^{-\frac{1}{p}}\max\{\|f\|_p,\ \|f\|_{p+1}^{\frac{p+1}{p}}\}$$

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An iterative interpolation argument $L^p - L^\infty$ ends the proof.

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- By the Trudinger inequality we obtain $e^{w^2/c} - 1 \in L^1(\mathbb{R})$.
- We end with the inequality

$$(e^w - 1)^2 \leq (e^c - 1)(e^{w^2/c} - 1)$$

that $u = e^w - 1 \in L^2(\mathbb{R})$.

Lemma

For every w, $c > 0$ it holds

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Develop the function $f(w) = (e^c - 1)(e^{w^2/c} - 1) - (e^w - 1)^2$ in its Taylor series and rearrange the terms as follows:

$$f(w) = \sum_{n=1}^{\infty} \frac{c^n}{n!} \sum_{k=1}^{\infty} \frac{w^{2k} c^{-k}}{k!} - \sum_{n=1}^{\infty} \frac{w^n}{n!} \sum_{k=1}^{\infty} \frac{w^k}{k!}$$

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Theorem

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- C^α regularity (for some $0 < \alpha < 1$) follows from [Athanasopoulos-Caffarelli'10]: the nonlinearity $\Phi(u)$ satisfies the nondegeneracy condition imposed in that paper.
- We write the equation (in a linear way) as

$$\partial_t u + \mu(-\Delta)^{1/2} u = -(-\Delta)^{1/2} F(u),$$

where $F(u) = \log(1+u) - \mu u - \log(1+u_0) + \mu u_0$, $\mu = 1/(1+u_0)$, $u_0 = u(x_0, t_0)$.

Smoothing effect

$L^\infty \to C^{1,\alpha}$

$$\begin{cases} \partial_t u + \mu(-\Delta)^{1/2} u = -(-\Delta)^{1/2} F(u) & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \end{cases}$$

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The solution can then be written as

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} P(x - x_1, \mu t) f(x_1)\, dx_1 \\ &\quad - \int_0^\infty \int_{\mathbb{R}} P(x - x_1, \mu(t - t_1))(-\Delta)^{1/2} F(u(x_1, t_1))\, dx_1 dt_1 \end{aligned}$$

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$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} P(x - x_1, \mu t) f(x_1)\, dx_1 \\ &- \int_0^\infty \int_{\mathbb{R}} (-\Delta)^{1/2} P(x - x_1, \mu(t - t_1)) F(u(x_1, t_1))\, dx_1 dt_1 \end{aligned}$$

Smoothing effect
$L^\infty \to C^{1,\alpha}$

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$F(u) = \log(\mu(1+u)) - \mu(u - u_0)$. $F(u_0) = F'(u_0) = 0$.
Therefore $F(u) \le (u - u_0)^2$. We use the doubling Hölder technique of [Caffarelli-Vasseur'10].

Further properties

Theorem

- Positivity: $u(x,t) > 0$ for every $x \in \mathbb{R}$, $t > 0$.
- Conservation law: $\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} f(x)\,dx$ for every $t \geq 0$.

$$H(v)(y) = \frac{1}{\pi}\text{P.V.}\int_{\mathbb{R}} \frac{v(s)}{y-s}\,ds$$

- $\widehat{H(v)}\,(\xi) = -i\,\text{sign}(\xi)\widehat{v}\,(\xi)$; $H^2 = -I$

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- $(-\Delta)^{1/2}(v)(y) = \frac{1}{\pi}\text{P.V.}\int_{\mathbb{R}} \frac{v(y)-v(s)}{|y-s|^2}\,ds = H(\partial_y v)(y)$
- $H(\partial_y v) = \partial_y H(v)$, whenever $v,\ \partial_y v \in L^p(\mathbb{R})$

$$\rightsquigarrow\ (-\Delta)^{1/2} = \partial_y H$$

A nonlocal transport equation
Change of variables

We put $(x, t, u) \mapsto (y, \tau, v)$ given by the Backlund type transform

$$y = \beta(x,t) = \int_0^x (1 + u(s,t))\, ds - c(t)\,, \qquad c'(t) = H(\Phi(u))(0,t)$$

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$$\begin{aligned} \widetilde{H}(v)(y,\tau) &= \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\Phi(u(s,t))}{\beta^{-1}(y,\tau) - s}\, ds \\ &= \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{v(z,\tau)}{\int_z^y e^{v(z,\tau) - v(\sigma,\tau)}\, d\sigma}\, dz\,. \end{aligned}$$

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Observe also that $y \in \mathbb{R}$, $\tau > 0$.

A nonlocal transport equation

We have then the transport equation

$$\partial_\tau v - \widetilde{H}(v)\partial_y v = -\partial_y \widetilde{H}(v)$$

with nonlocal velocity and viscosity.

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see [Constantin-Lax-Majda'85, Baker-Li-Morlet'96, Córdoba-Córdoba-Fontelos'05]. 1-D model for 2-D quasigeostrophic equation and 3-D Navier-Stokes equation.

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For this latter equation it is known that, for initial values in H^1:

- without viscosity, blow-up (in C^1) can happen;
- viscosity (critical) prevents blow-up provided the initial datum is small.

A nonlocal transport equation

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$$\int_{\mathbb{R}} |\partial_y v(y)|^2\, dy = \int_{\mathbb{R}} \frac{1}{1 + u(x)} |\partial_x \Phi(u(x))|^2\, dx \leq \int_{\mathbb{R}} |\partial_x \Phi(u(x))|^2\, dx$$

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Therefore,

$$v \in L^1(\mathbb{R}) \;\Leftrightarrow\; u \in \mathcal{X}, \qquad \Phi(u) \in H^1(\mathbb{R}) \;\Rightarrow\; v \in H^1(\mathbb{R})$$

A nonlocal transport equation

Theorem

- For every $v_0 \in L^1(\mathbb{R})$, $v_0 \geq 0$, there exists a unique global in time solution with initial value v_0.

- L^1-L^∞ smoothing effect (for τ small):

$$\|v(\cdot,\tau)\|_\infty \leq \tau^{-1/2}\|v_0\|_1^{1/2}.$$

- Decay (for τ large):

$$\|v(\cdot,\tau)\|_\infty \leq \tau^{-3/4}\|v_0\|_1^{3/4}.$$

- Regularity: $v \in C^{1,\alpha}(\mathbb{R}\times(0,\infty))$ for every $0<\alpha<1$.
- Positivity: $v(y,\tau) > 0$ for every $y \in \mathbb{R}$, $\tau > 0$.
- Conservation law: $\displaystyle\int_{\mathbb{R}} [1-e^{-v(y,\tau)}]\,dy = \int_{\mathbb{R}} [1-e^{-v_0(y)}]\,dy$ for every $\tau \geq 0$.

Open problems/questions

- Further regularity (C^∞)
- Other values of N and σ (relevant perhaps $N = 1 < \sigma < 2$)
- The parabolic equation obtained from the transport equation

$$\partial_t u = -\widetilde{\Lambda}\Phi(u)$$

Thanks!!!

gràcies!!